A Unified Framework for Schedule and Storage Optimization

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http://compiler.lcs.mit.edu/aov

Motivating Example

```
for i = 1 to n
  for j = 1 to n
    A[j] = f(A[j], A[j-2])
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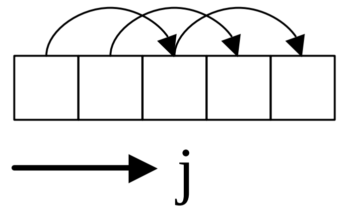

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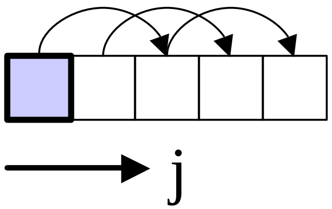

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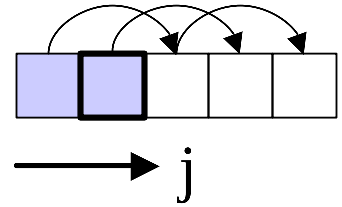

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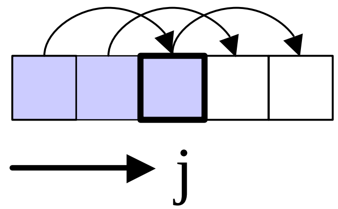

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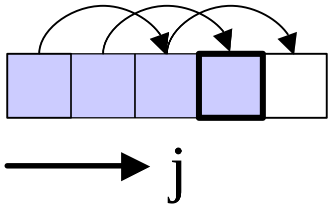

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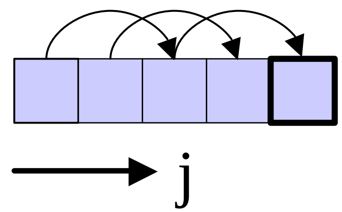

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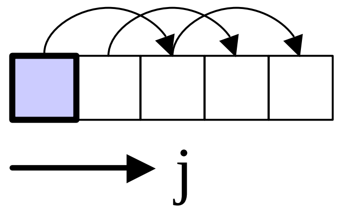

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$(i, j) = (2, 1)$

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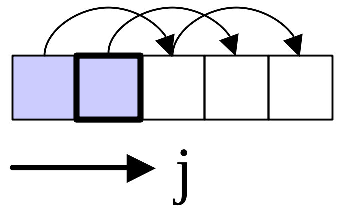

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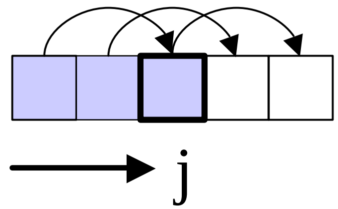

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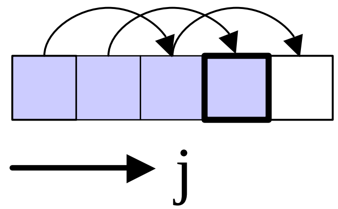

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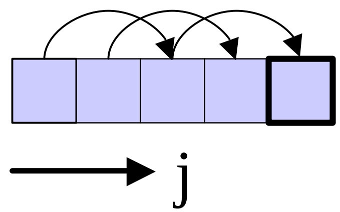

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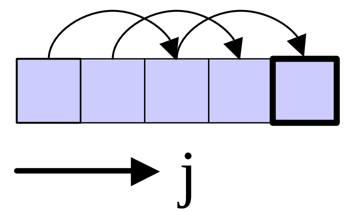

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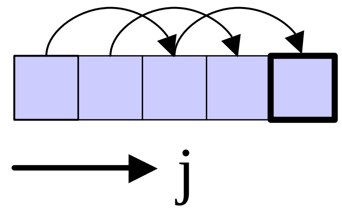

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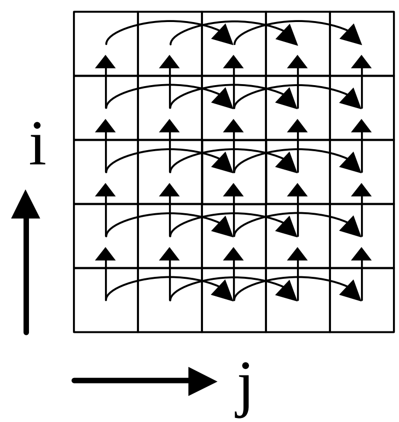

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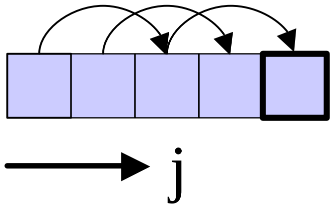

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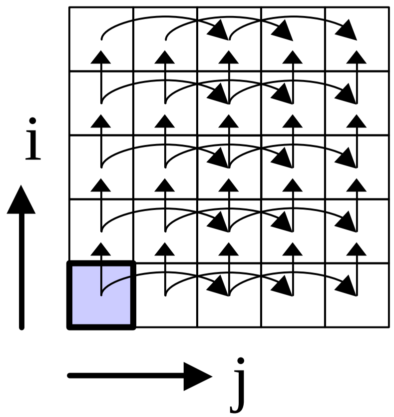

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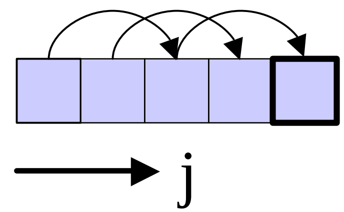

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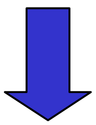

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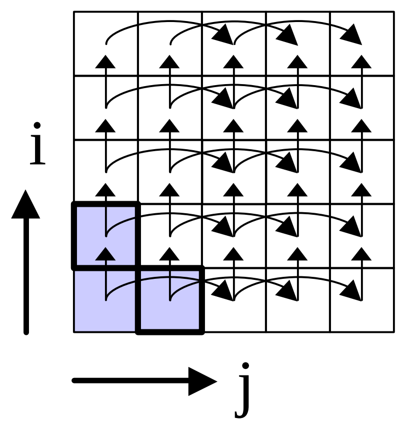

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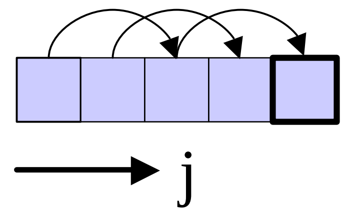

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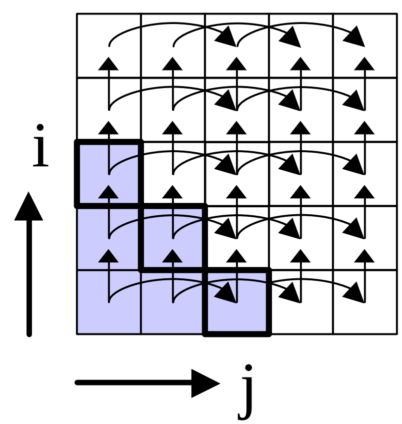

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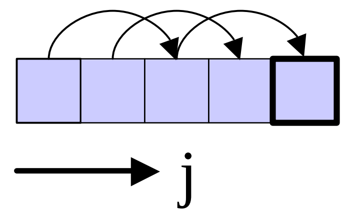

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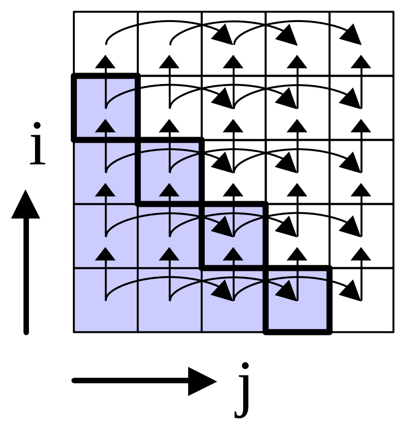

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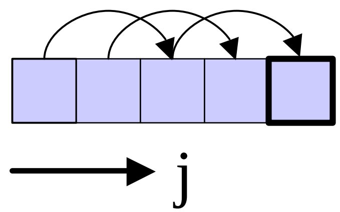

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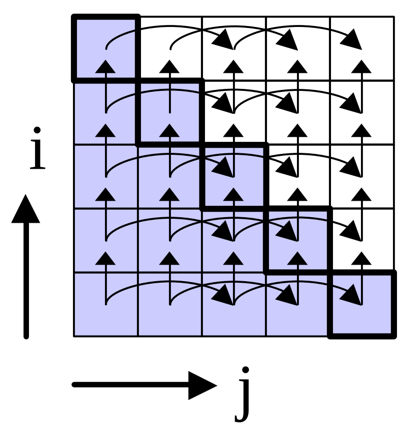

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$(i, j) = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$

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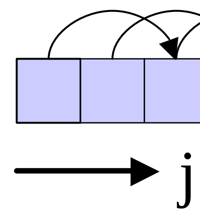

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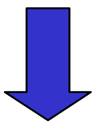

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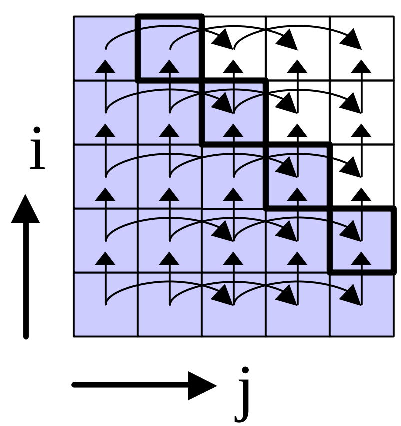

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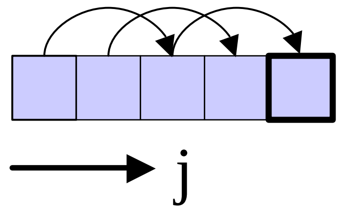

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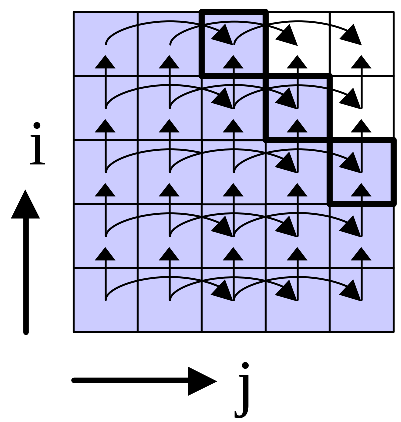

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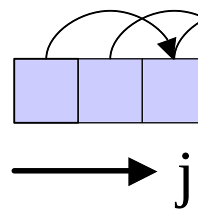

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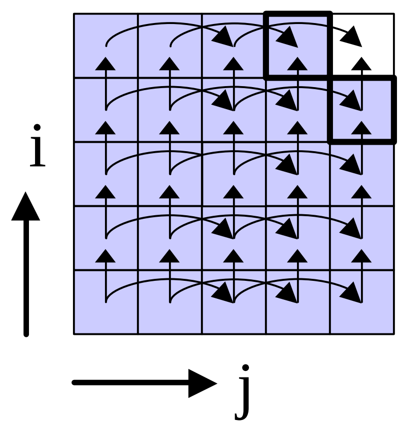

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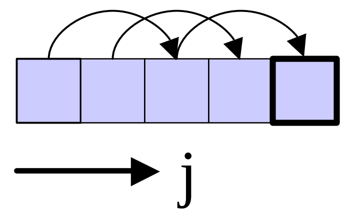

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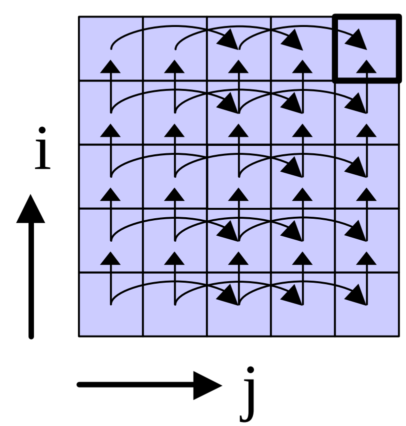
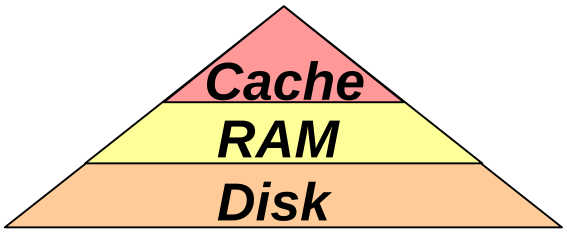


t = 9

(i, j) = (5, 5)

Parallelism/Storage Tradeoff

- Increasing storage can enable parallelism
  - But storage can be expensive



- Phase ordering problem
  - Optimizing for storage restricts parallelism
  - Maximizing parallelism restricts storage options
  - Too complex to consider all combinations

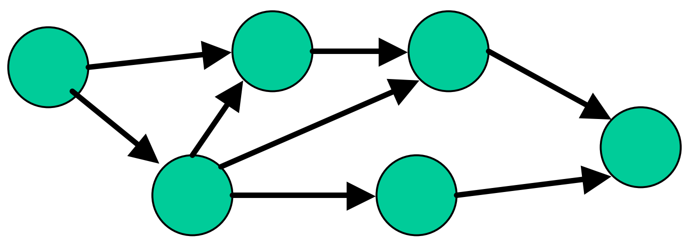
➔ Need efficient framework to integrate schedule and storage optimization

Outline

- Abstract problem
- Simplifications
- Concrete problem
- Solution Method
- Conclusions

Abstract Problem

- Given DAG of dependent operations



  - Must execute producers before consumers
  - Must store a value until all consumers execute
- Two parameters control execution:
  1. A scheduling function θ
     - Maps each operation to execution time
     - Parallelism is implicit
  2. A fully associative store of size m

Abstract Problem

- We can ask three questions:

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Abstract Problem

- We can ask three questions:
  1. Given θ, what is the smallest m?
  2. Given m, what is the "best" θ?
  3. What is the smallest m that is valid for all legal θ?

- Two parameters control execution:
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Simplifying the Schedule

- Real programs aren't DAG's
  - Dependence graph is parameterized by loops
  - Too many nodes to schedule
    - Size could even be unknown (symbolic constants)
- Use classical solution: affine schedules
  - Each statement has a scheduling function θ
  - Each θ is an affine function of the enclosing loop counters and symbolic constants
  - To simplify talk, ignore symbolic constants:

$$\theta(\vec{i}) = \vec{\beta} \cdot \vec{i}$$

Simplifying the Storage Mapping

- Programs use arrays, not associative maps
  - If size decreases, need to specify which elements are mapped to the same location

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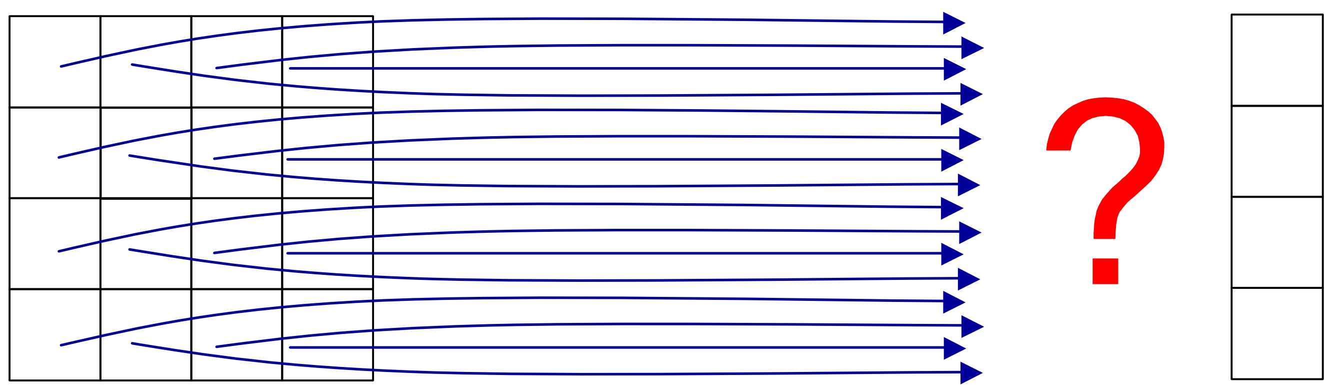
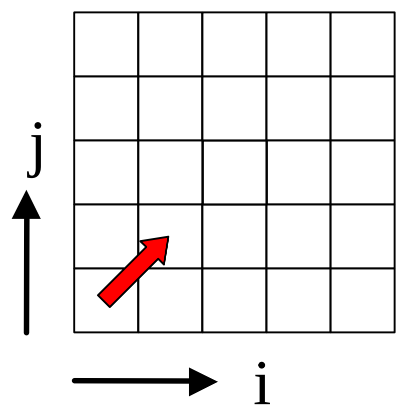


Occupancy Vectors (Strout et al.)

- Specifies unit of overwriting within an array
- Locations collapsed if separated by a multiple of $\vec{v}$

$\vec{v} = (1, 1)$



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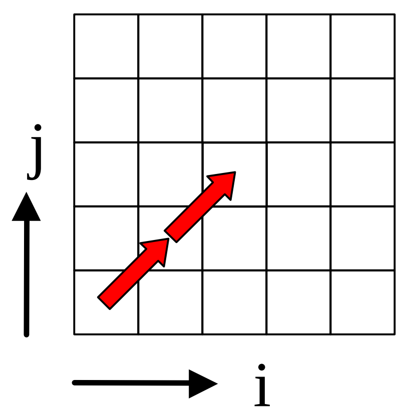

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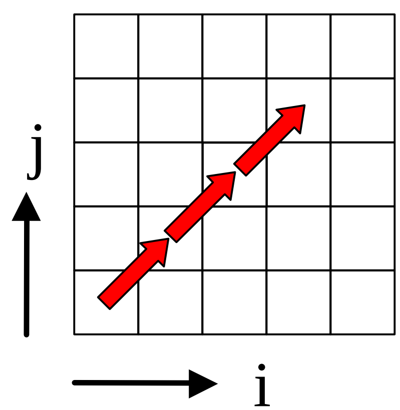

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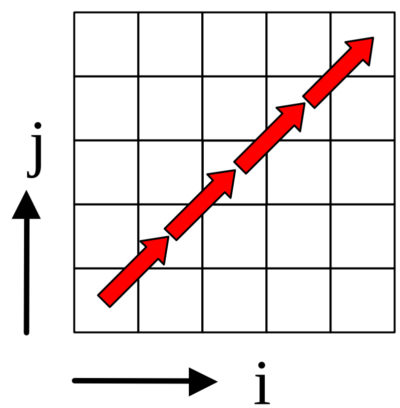

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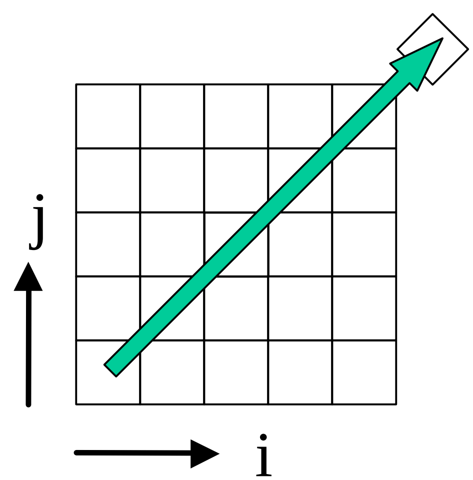

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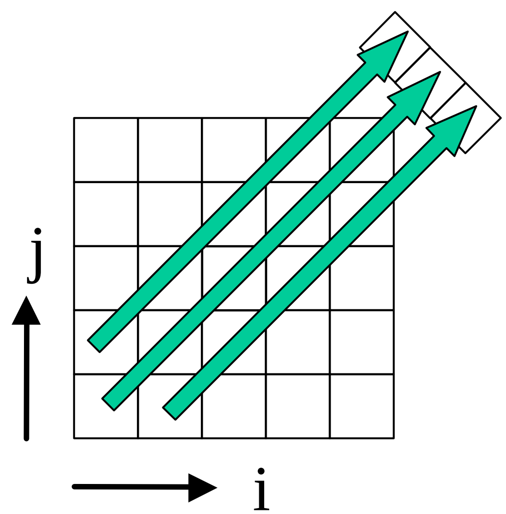

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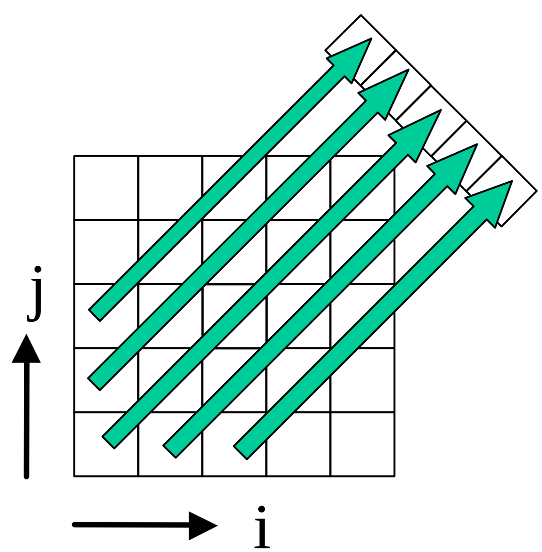

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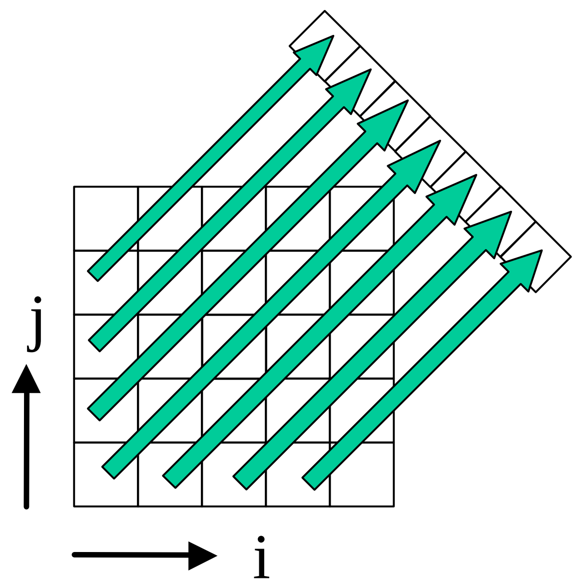

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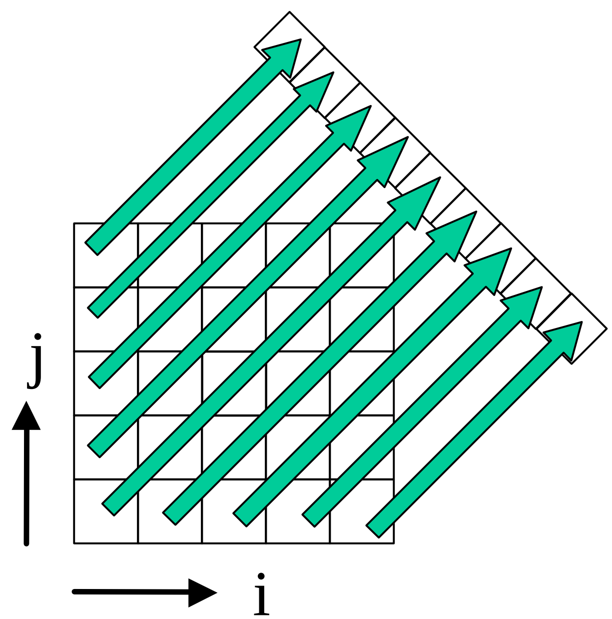

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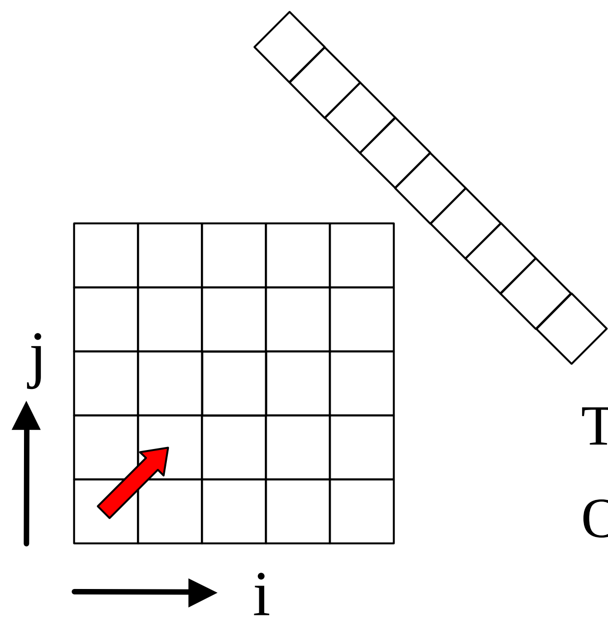

$\text{Transformed}: \sqrt{2}n$

$\text{Original}: n^2$

Occupancy Vectors (Strout et al.)

- For a given schedule, $\vec{v}$ is valid if semantics are unchanged using transformed array
- Shorter vectors require less storage

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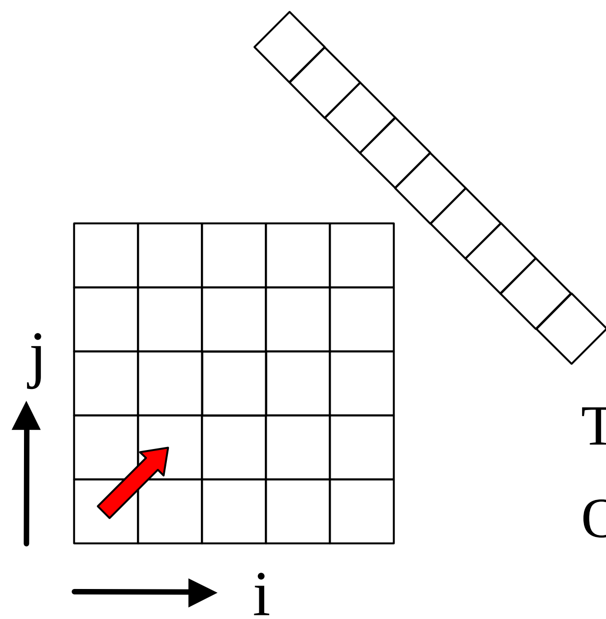

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Outline

- Abstract problem
- Simplifications
- Concrete problem
- Solution Method
- Conclusions

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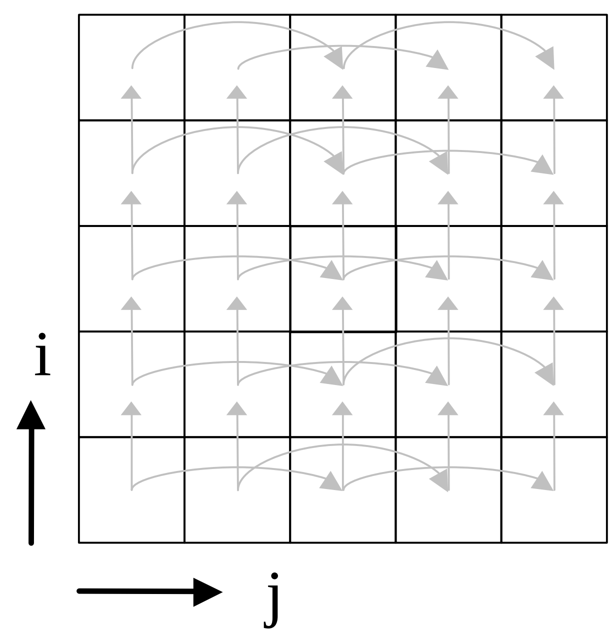

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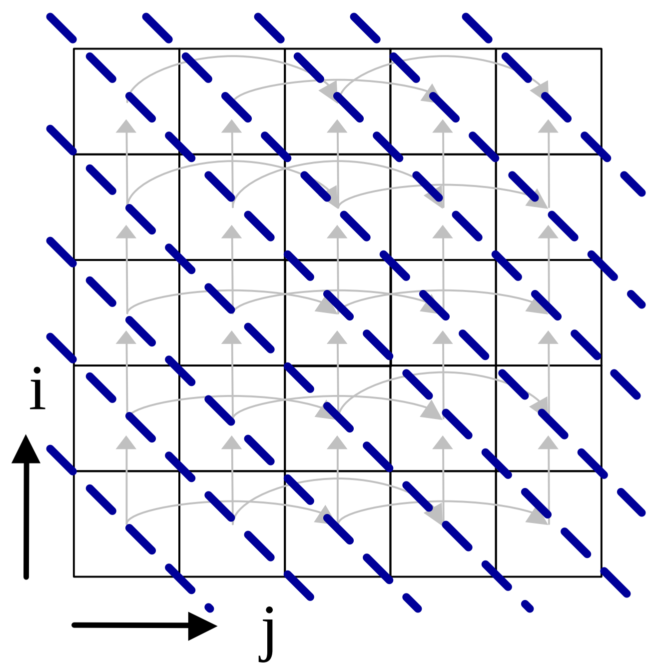

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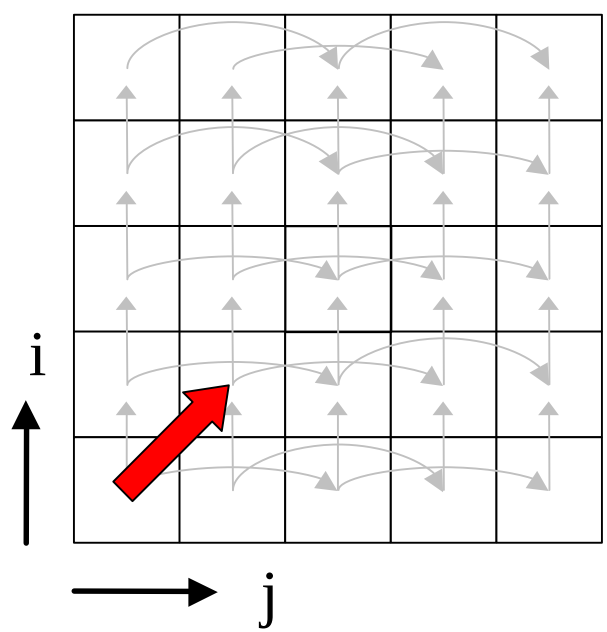

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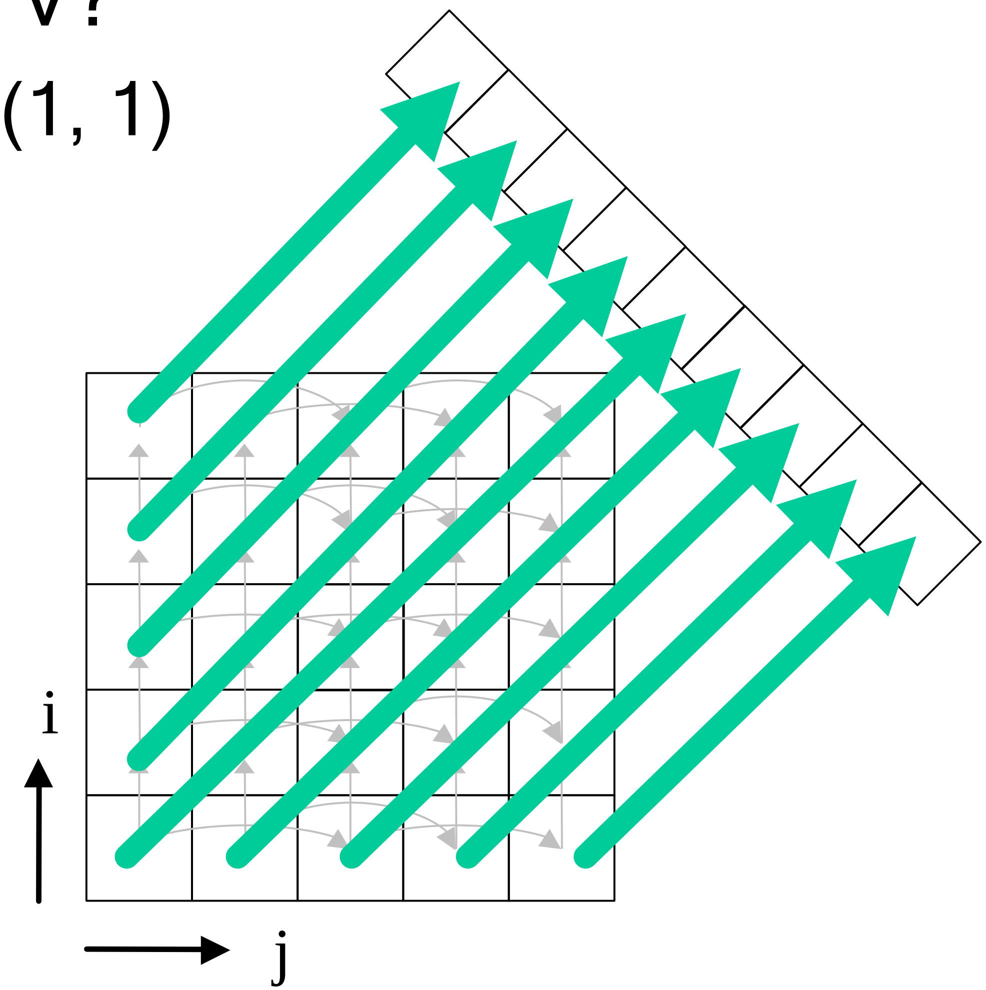

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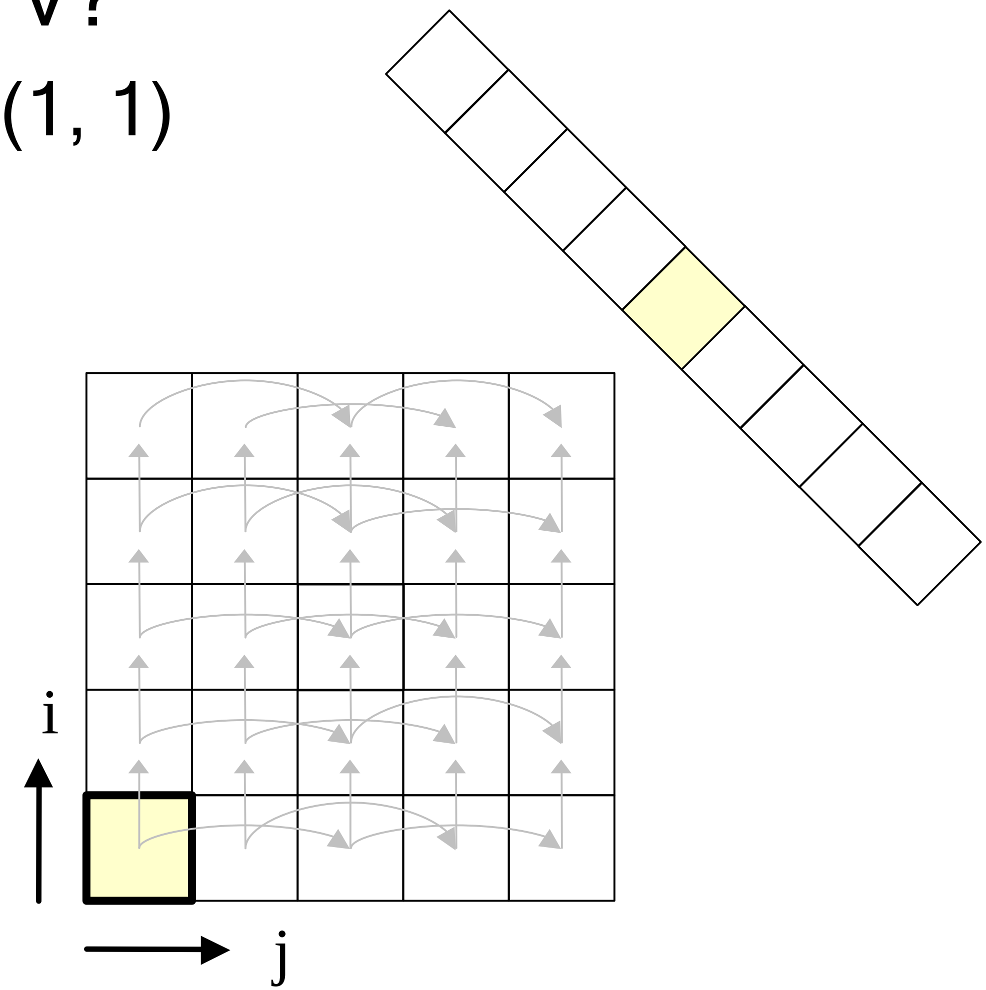

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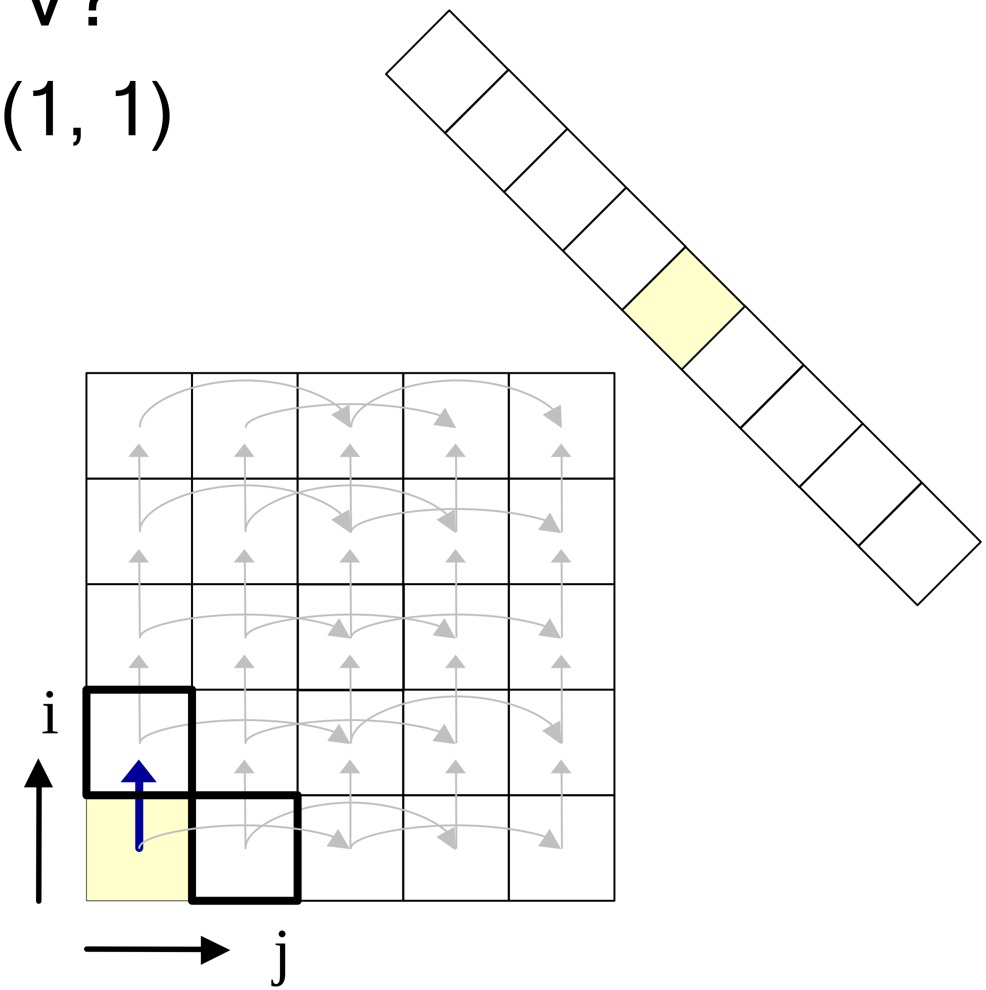

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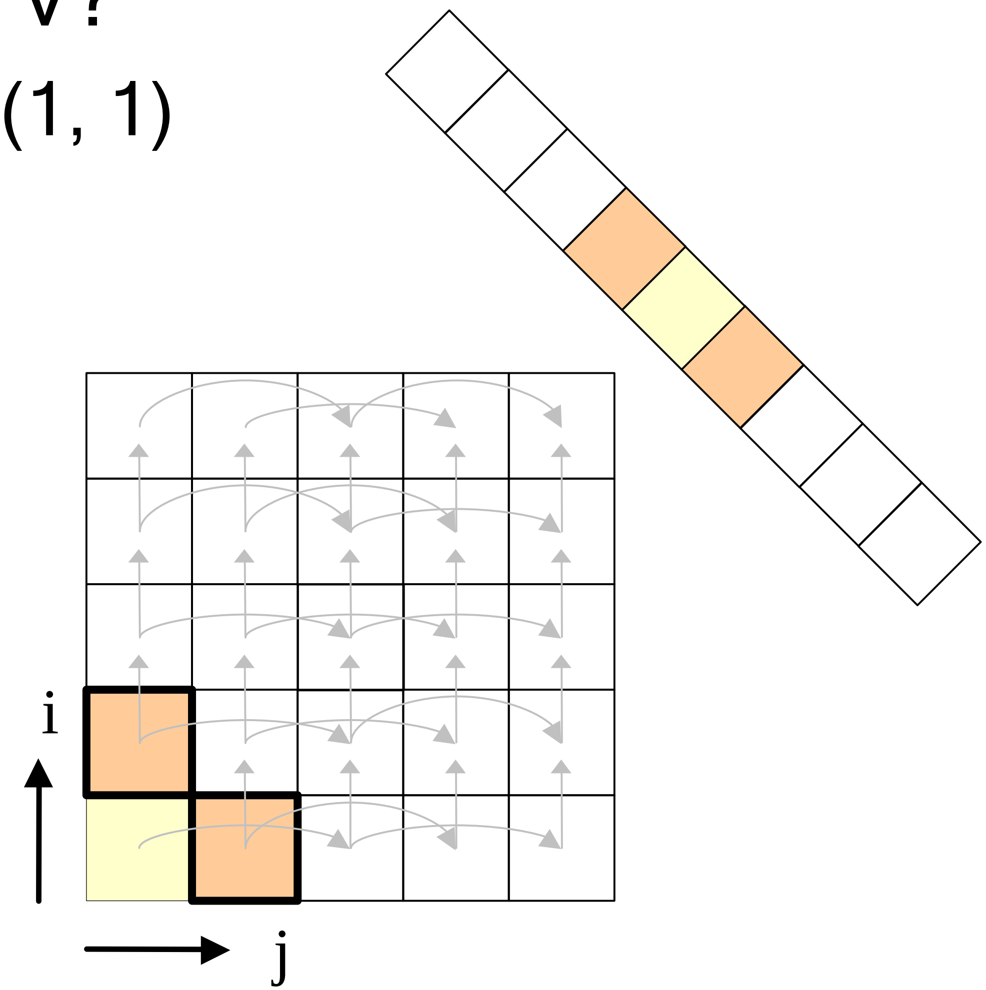

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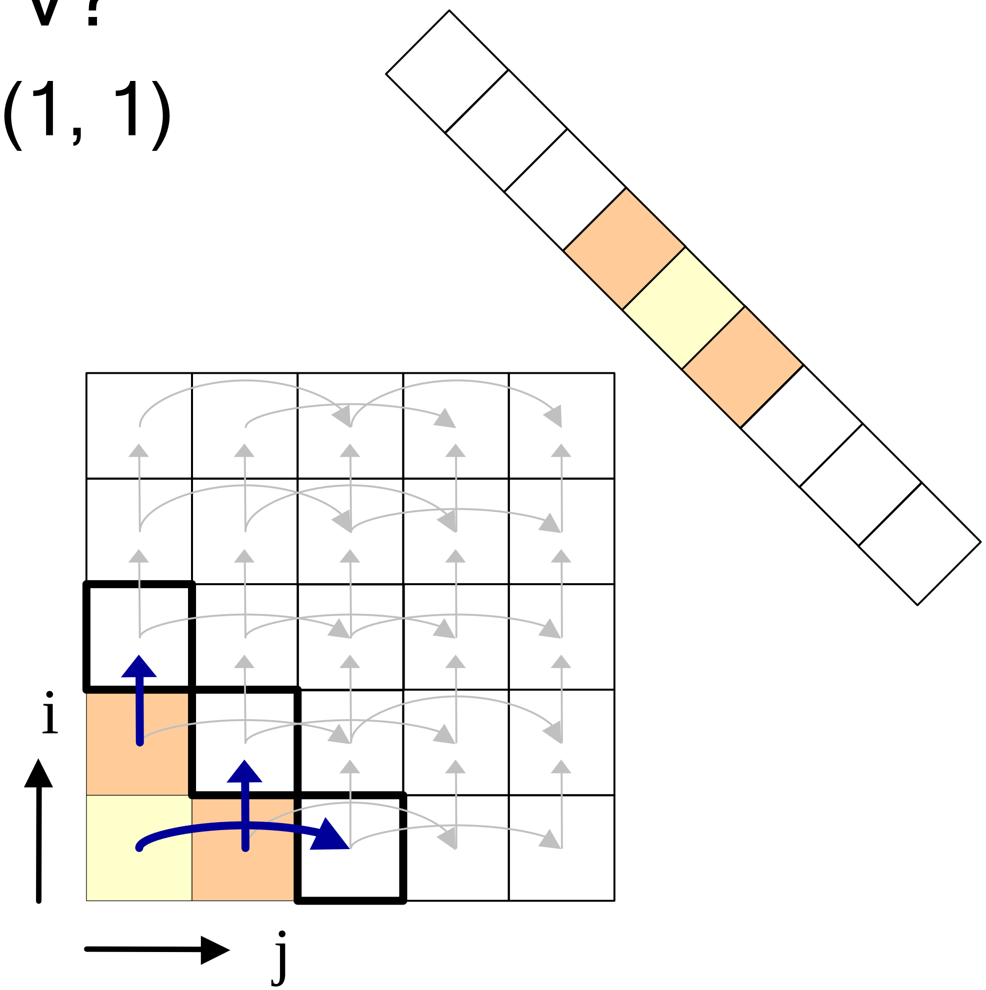

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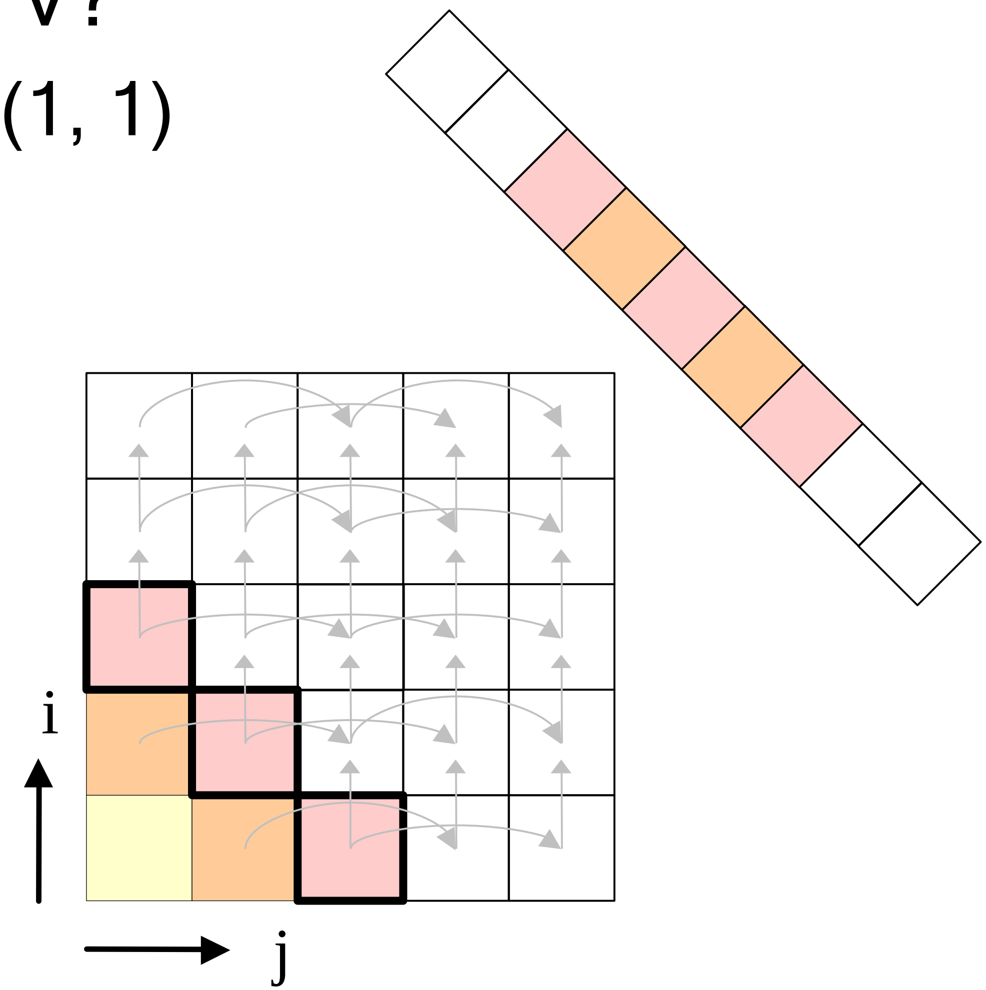

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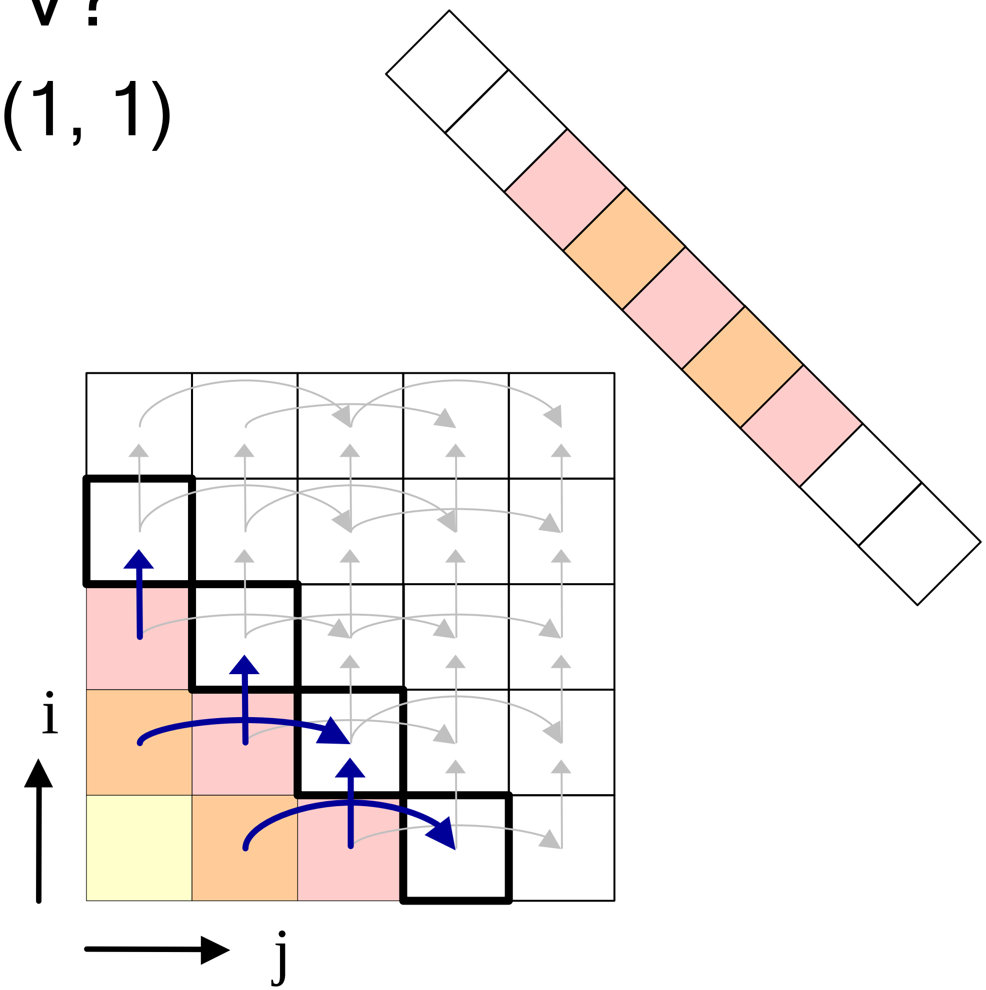

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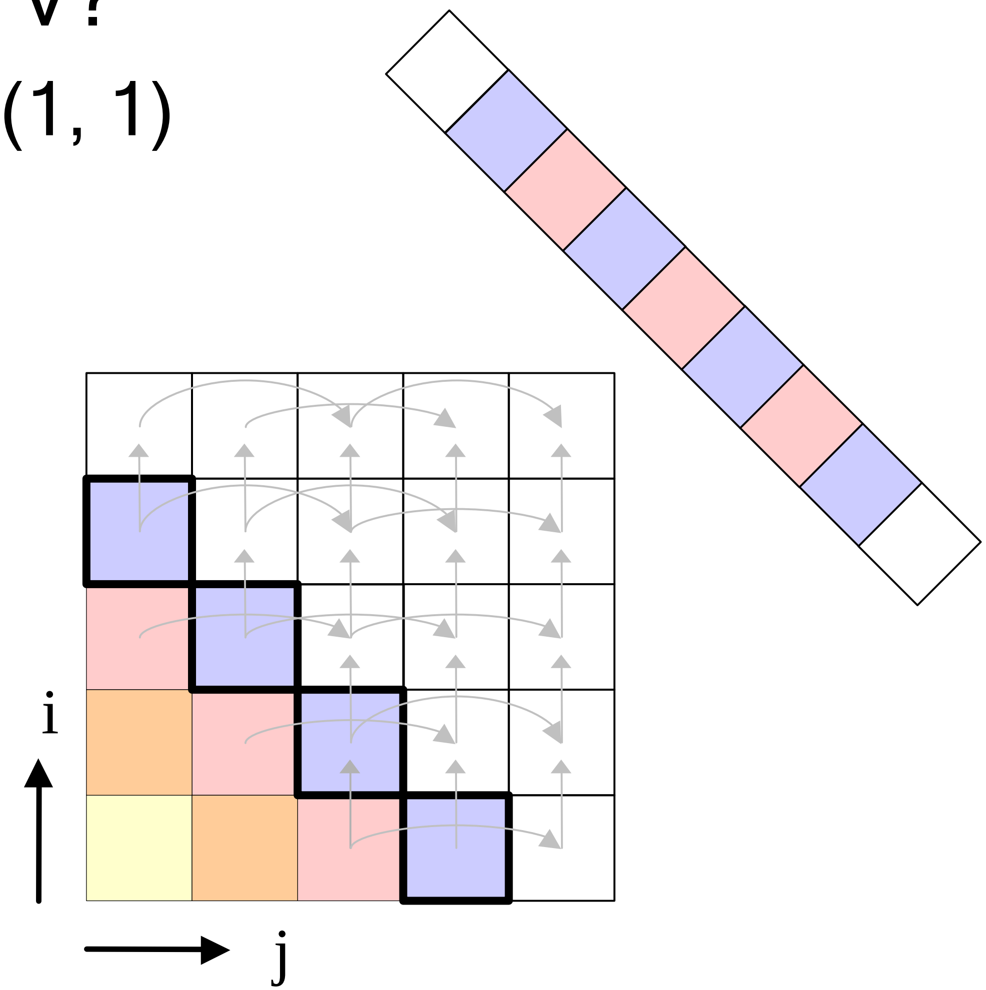

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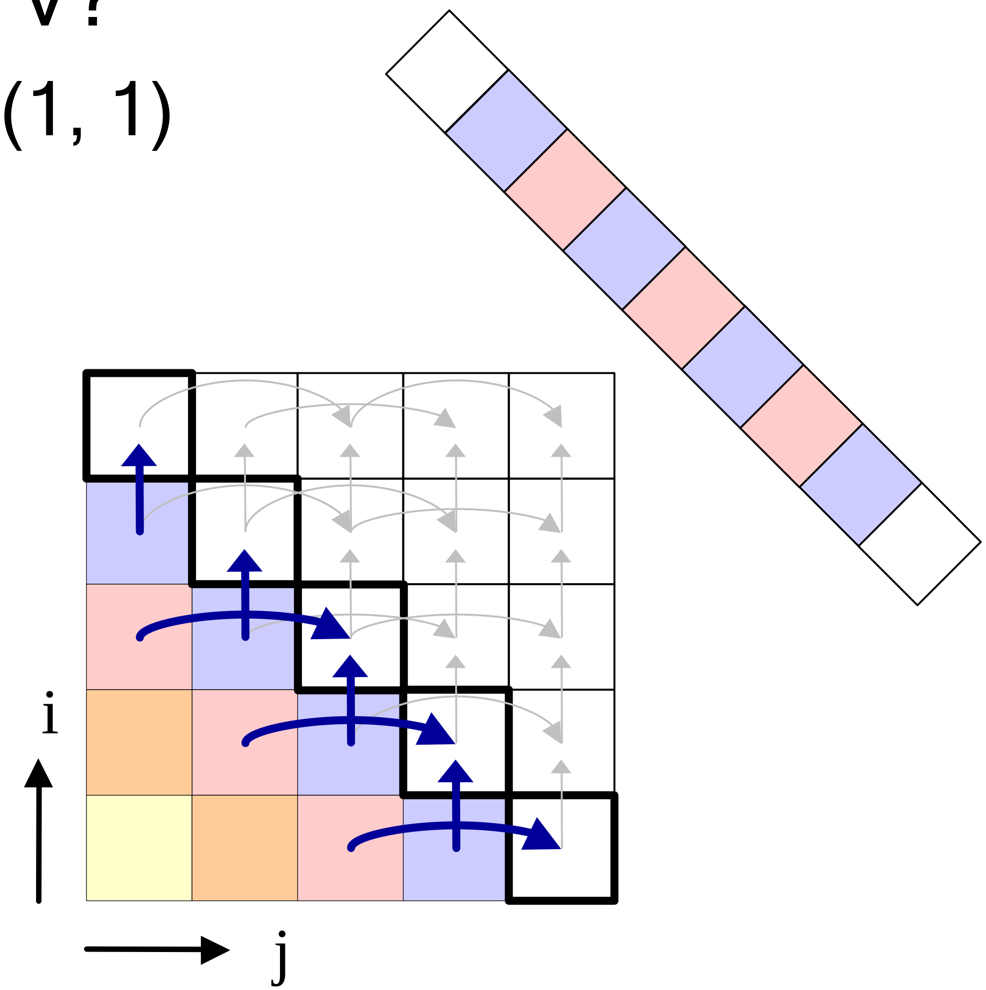

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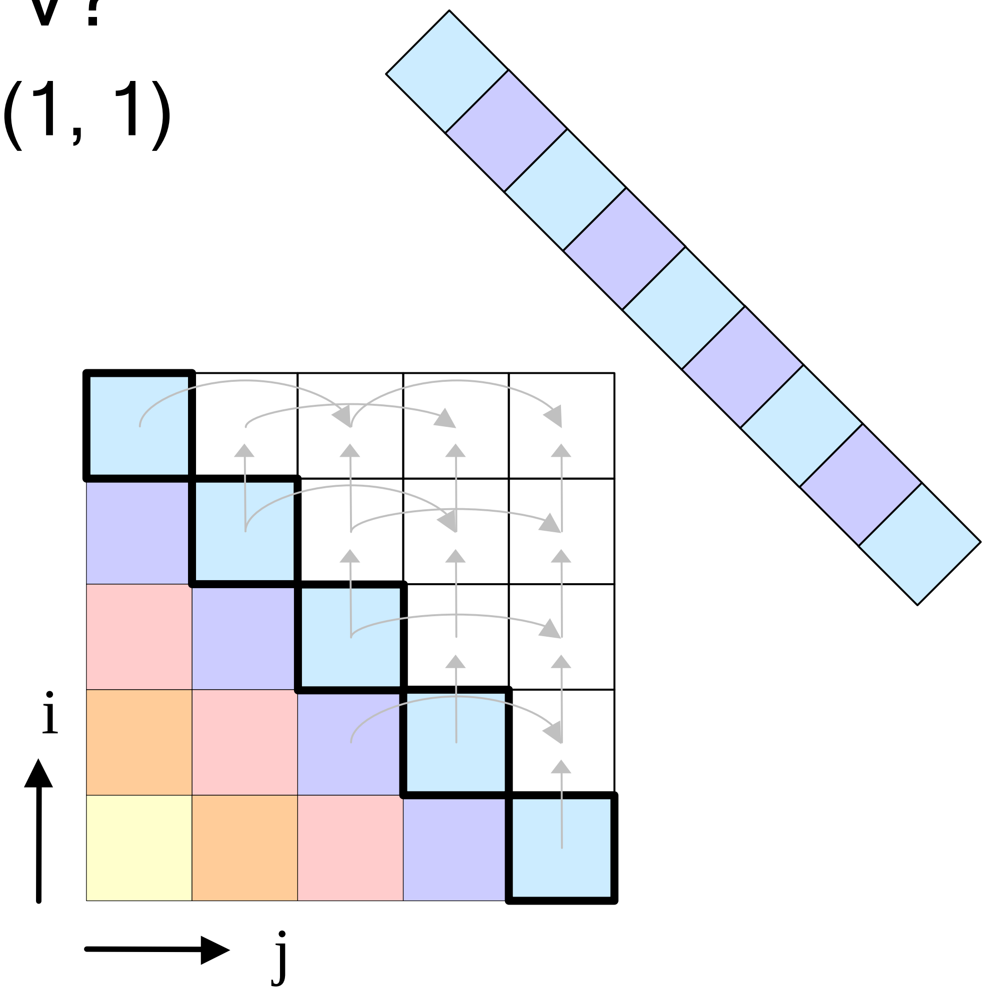

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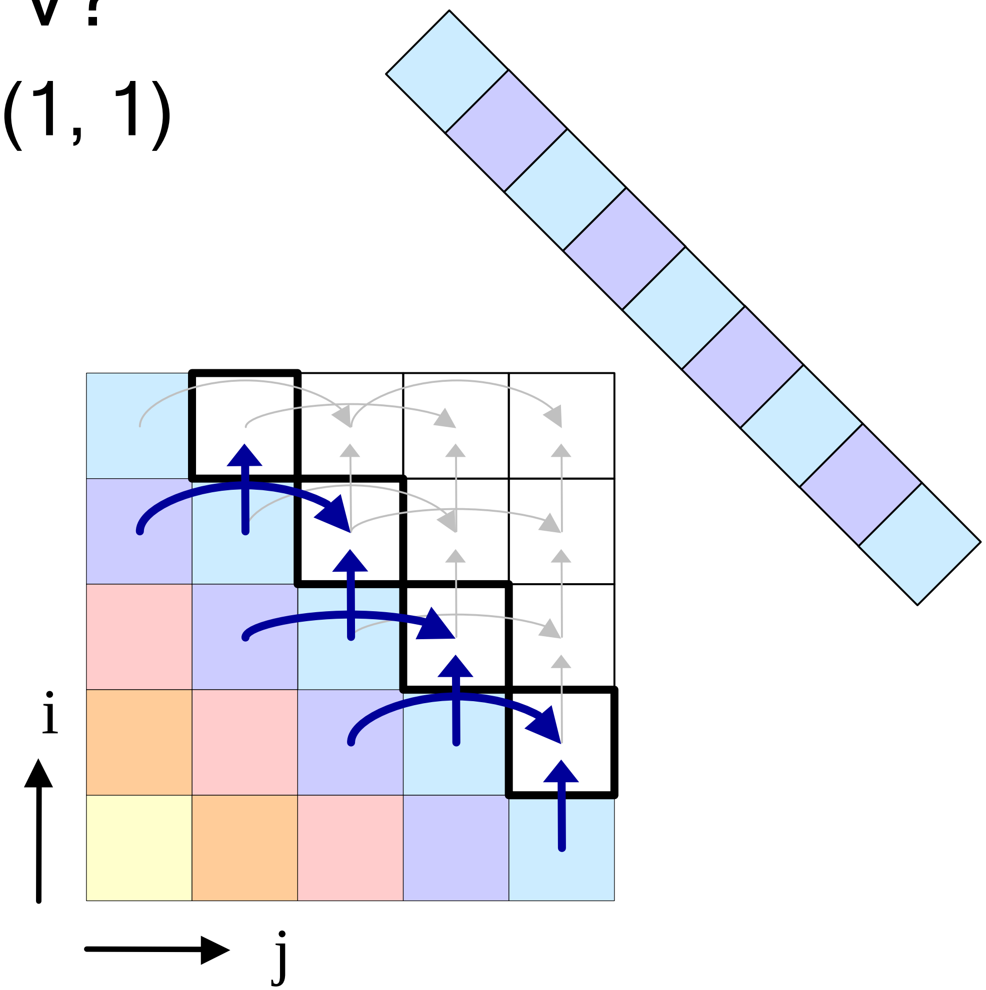

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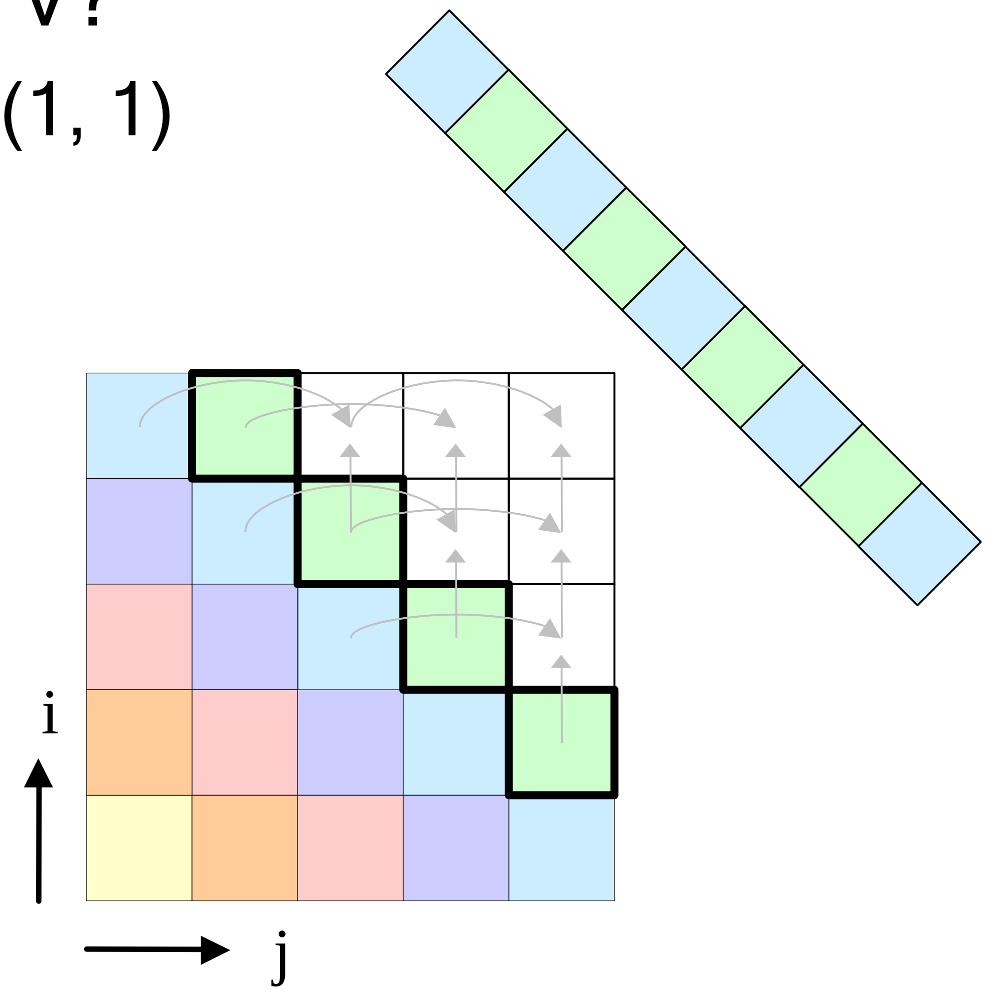

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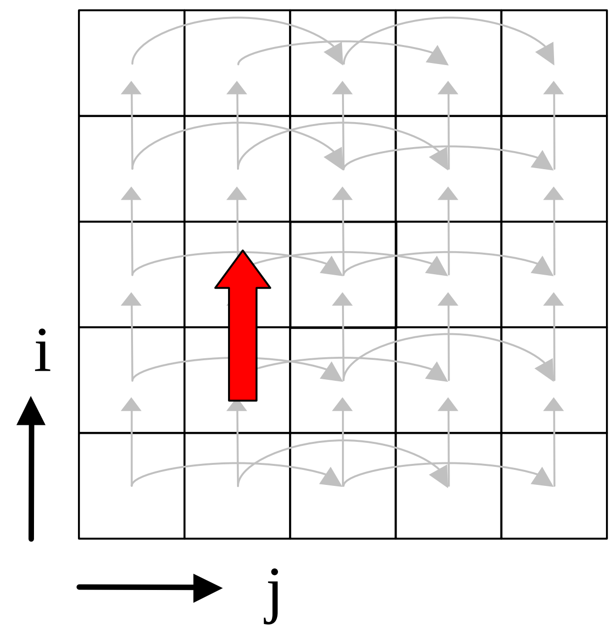

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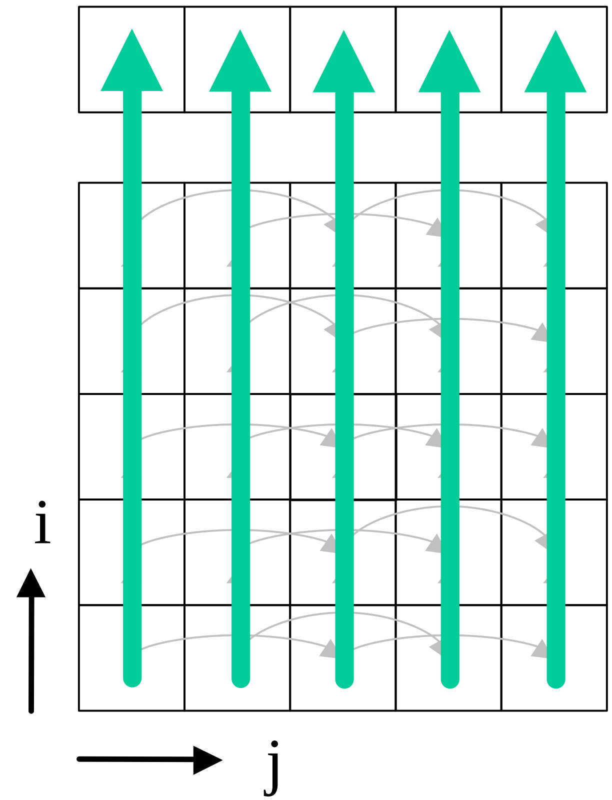

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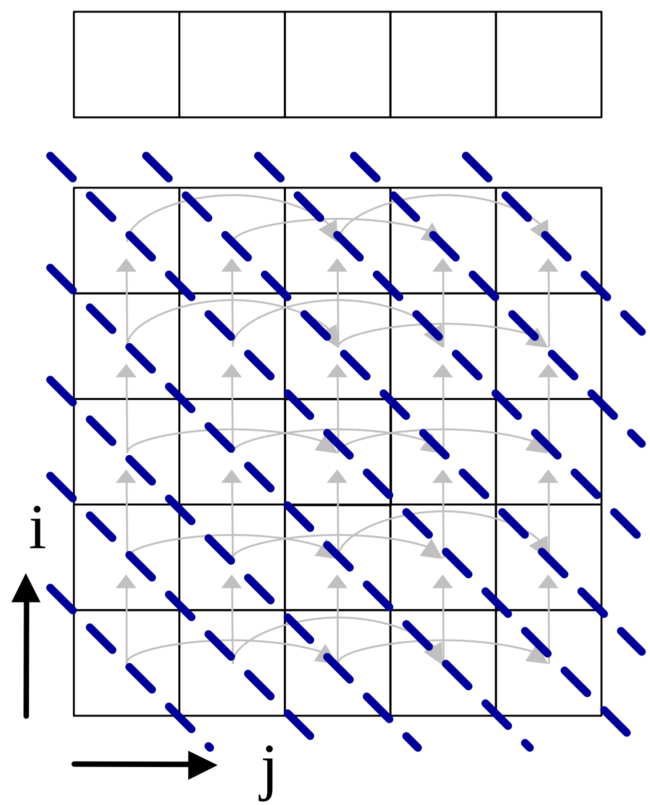

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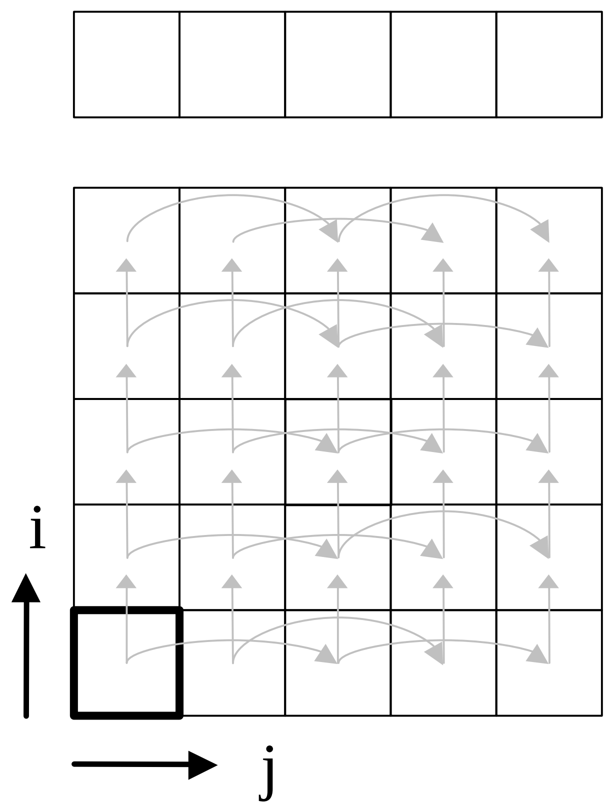

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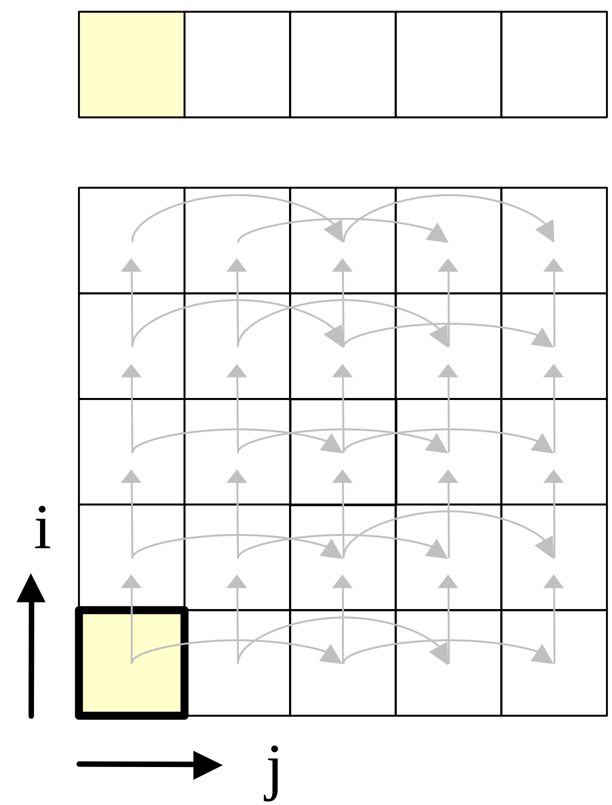

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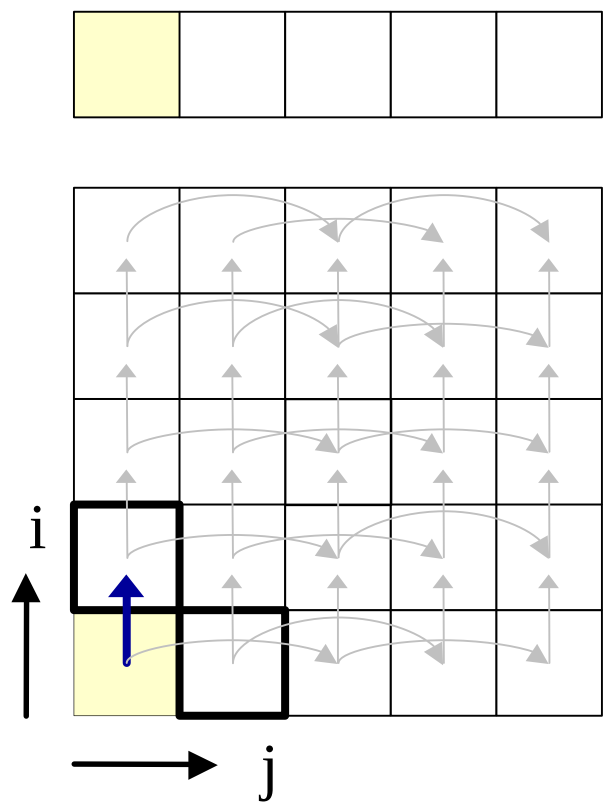

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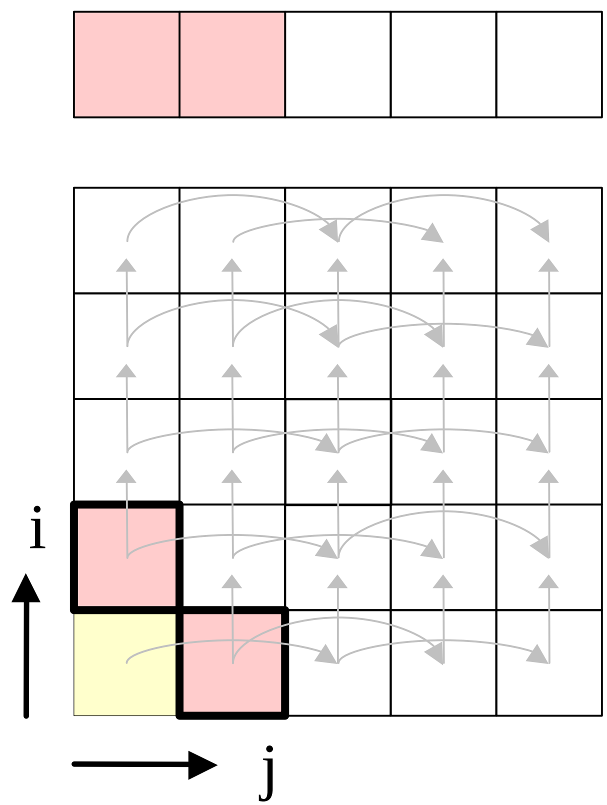

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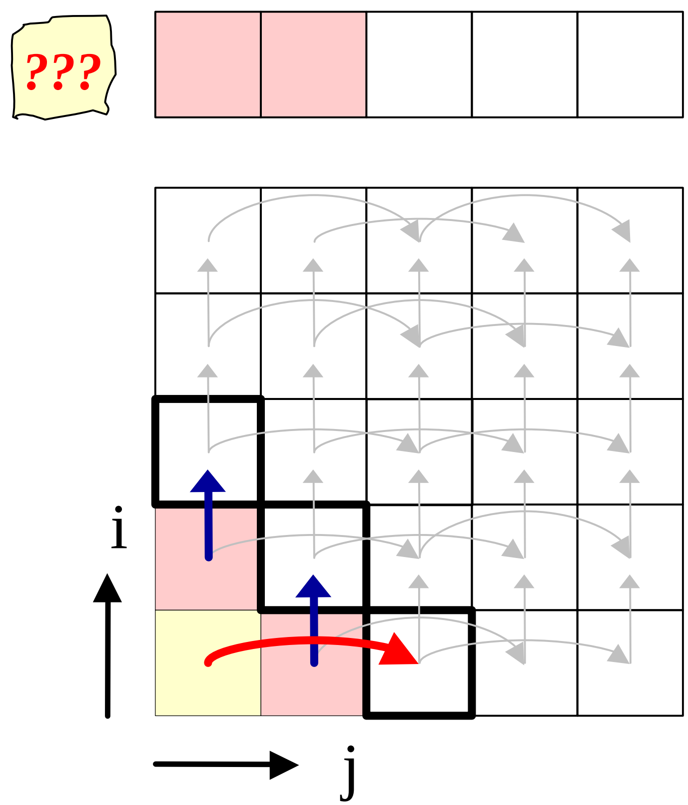

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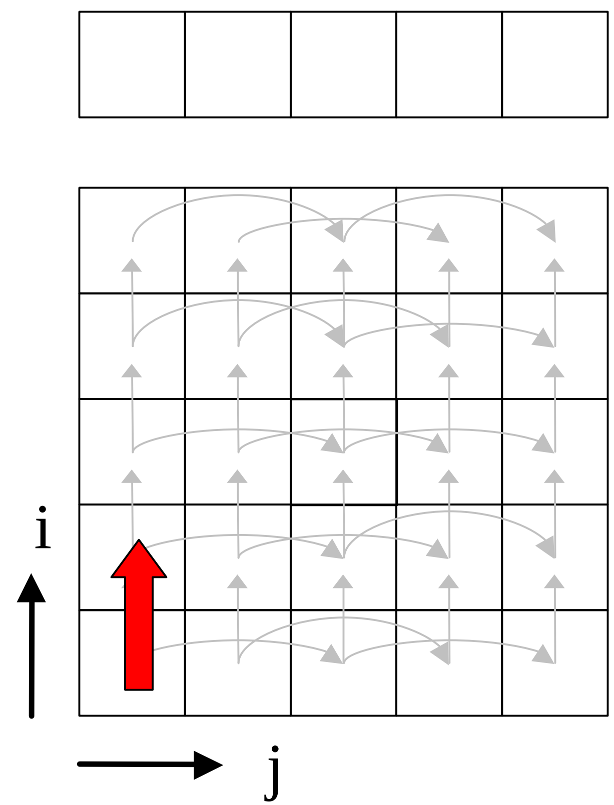

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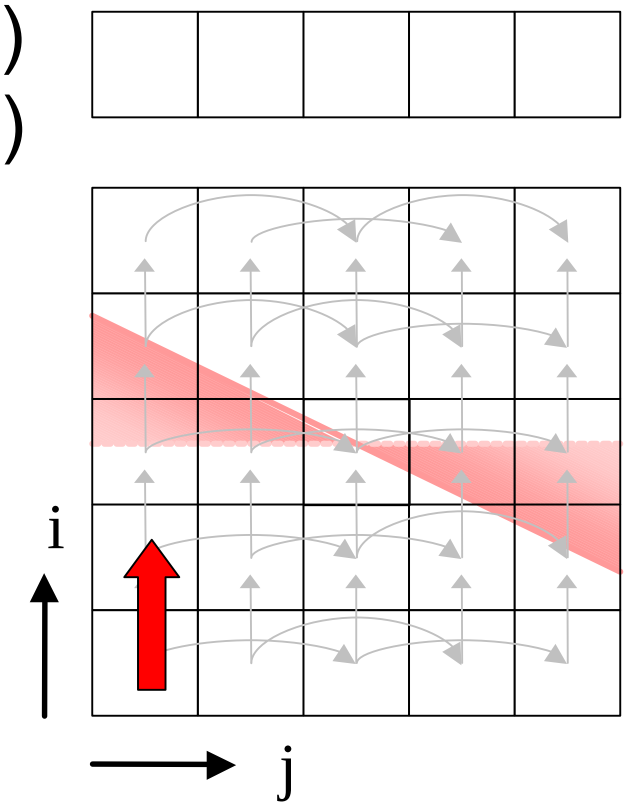

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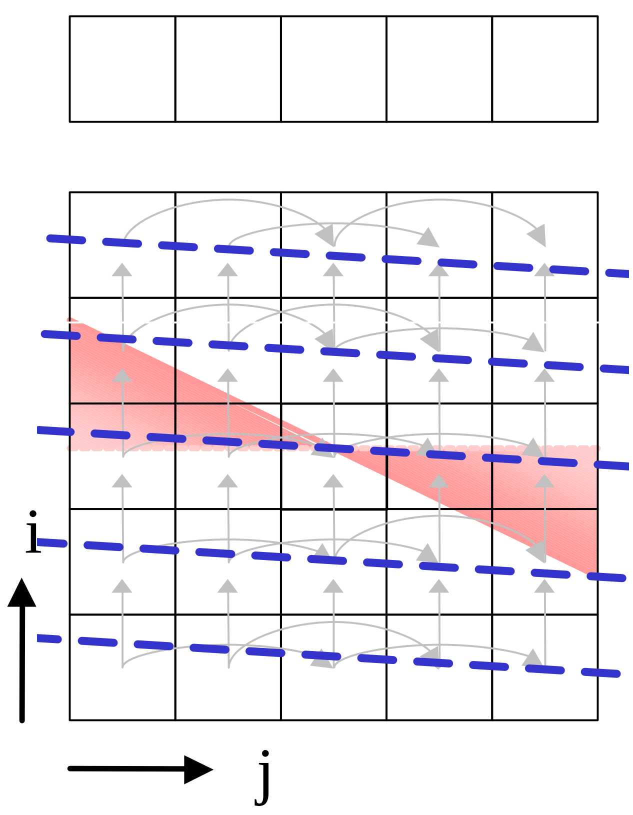

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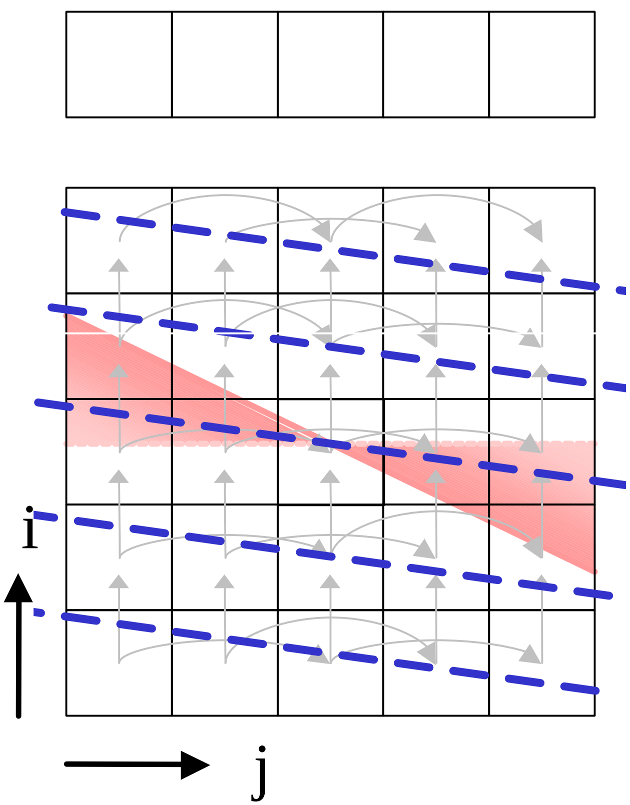

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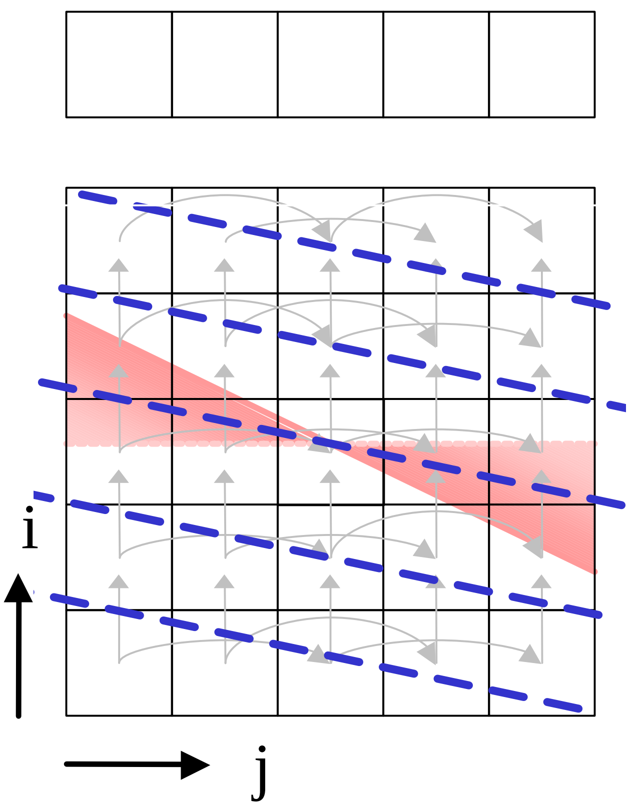

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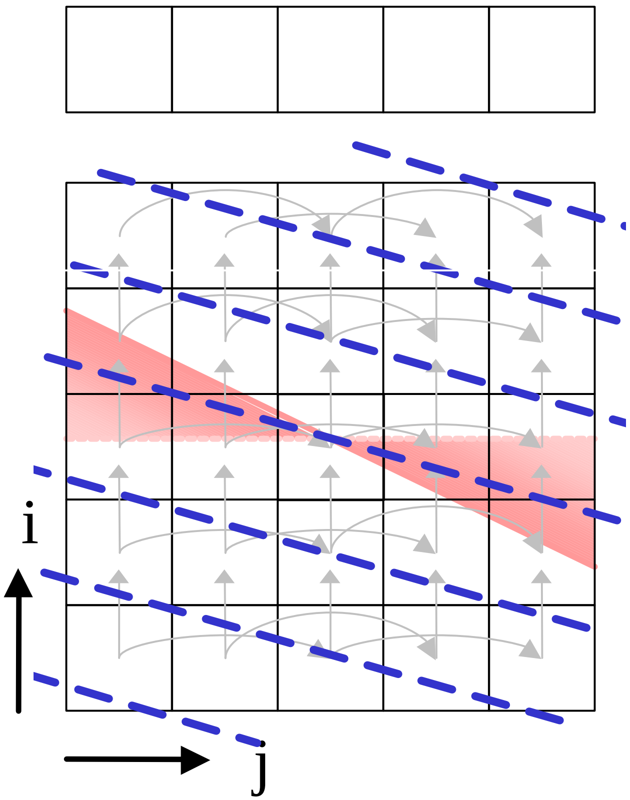

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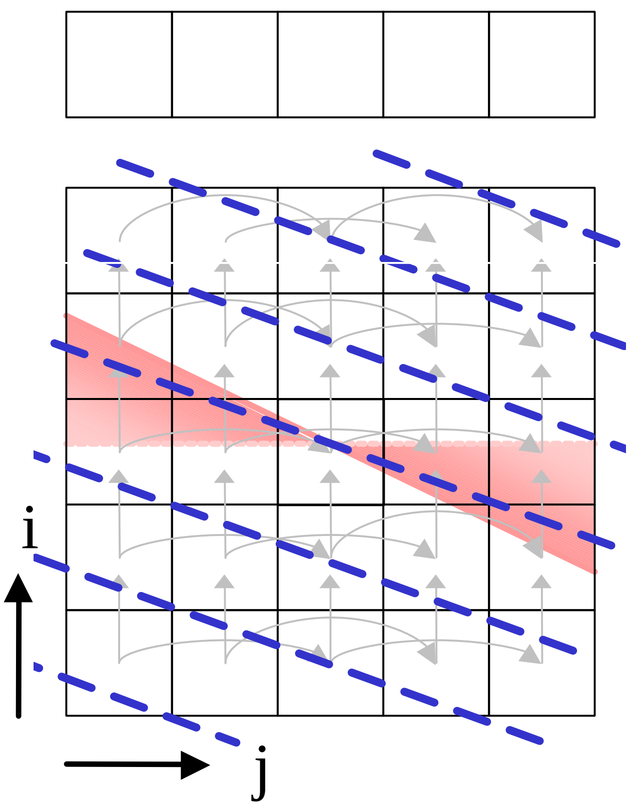

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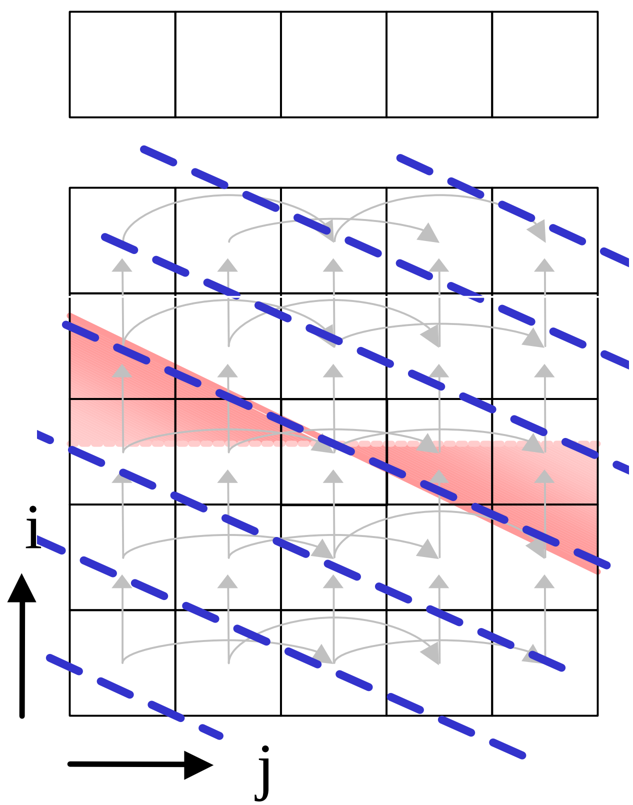

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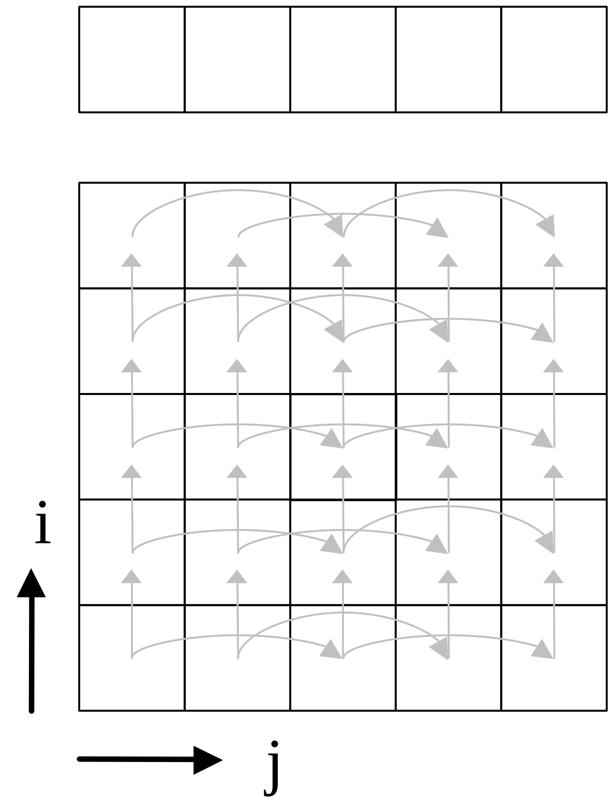

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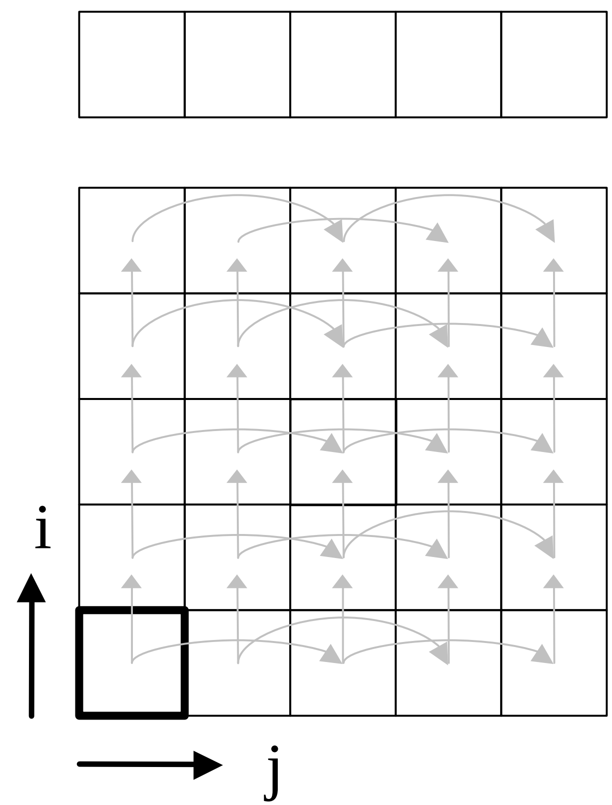

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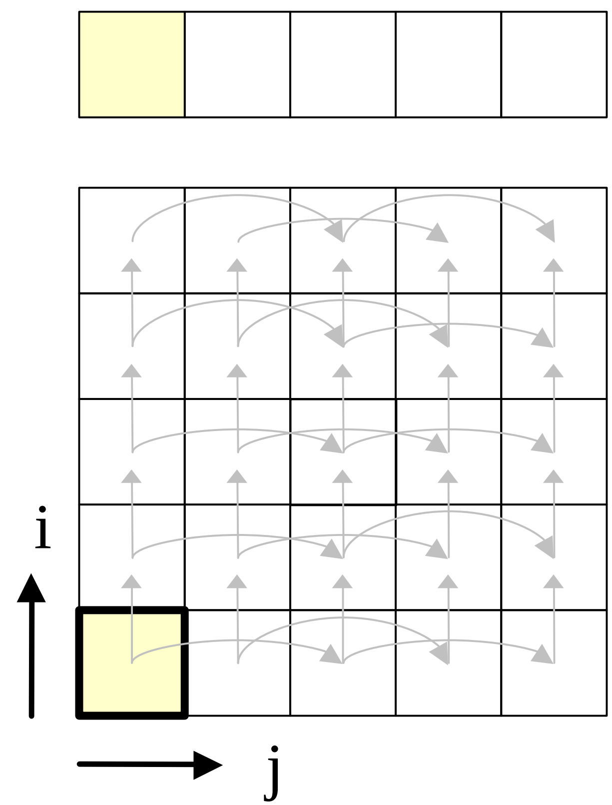

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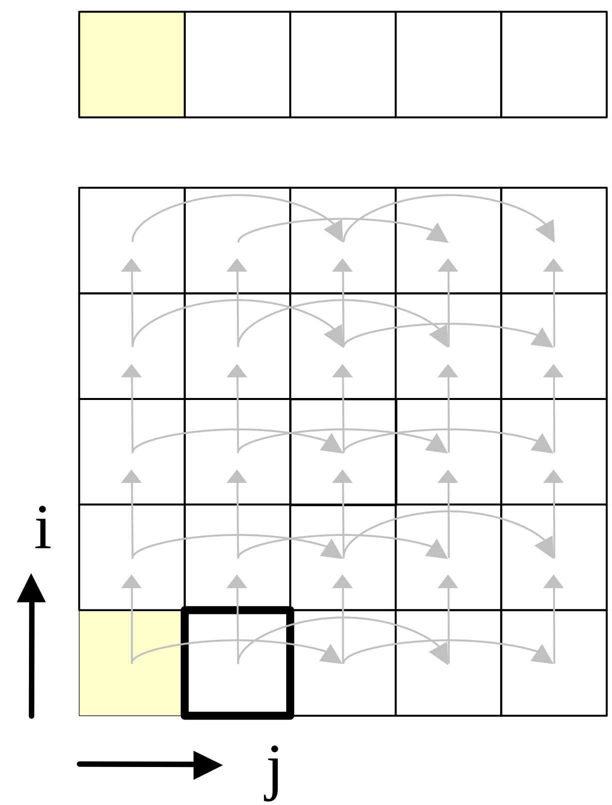

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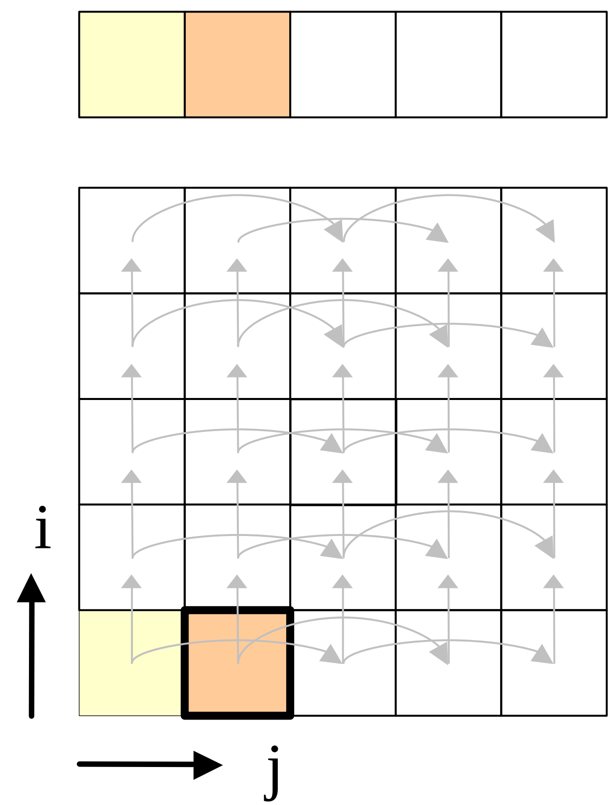

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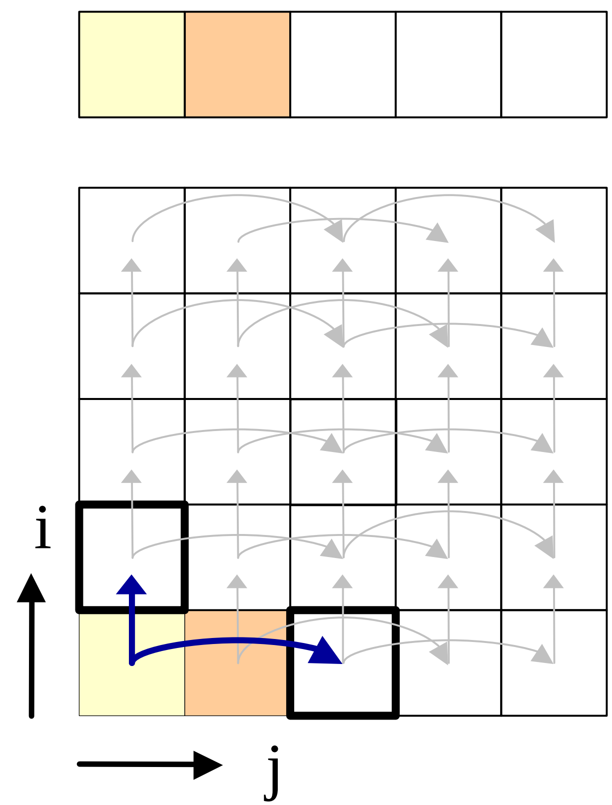

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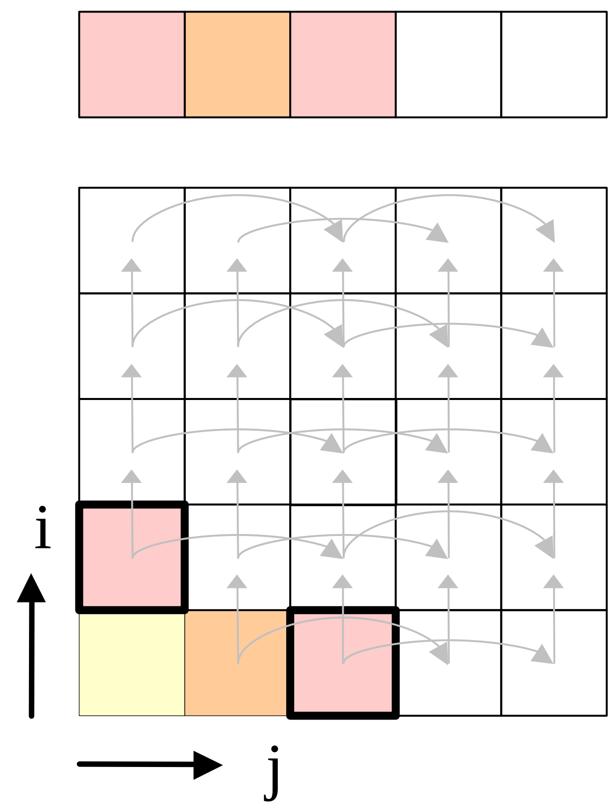

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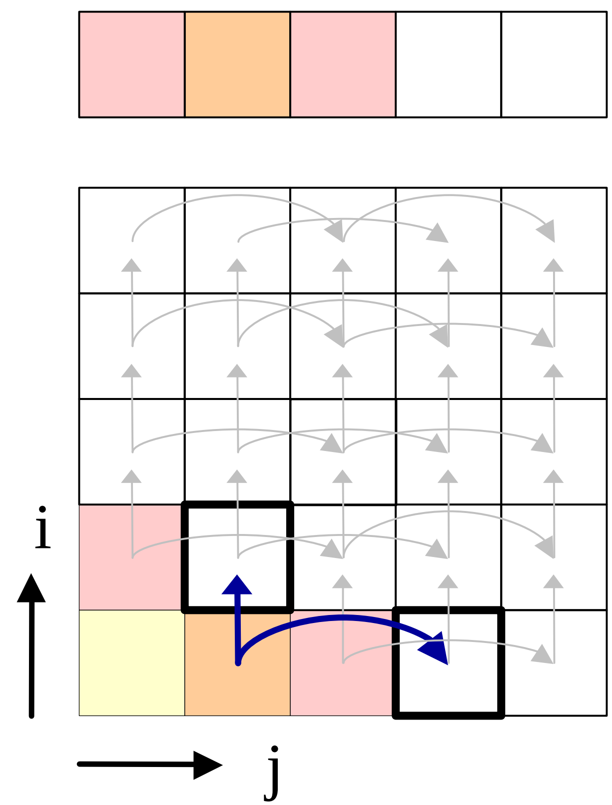

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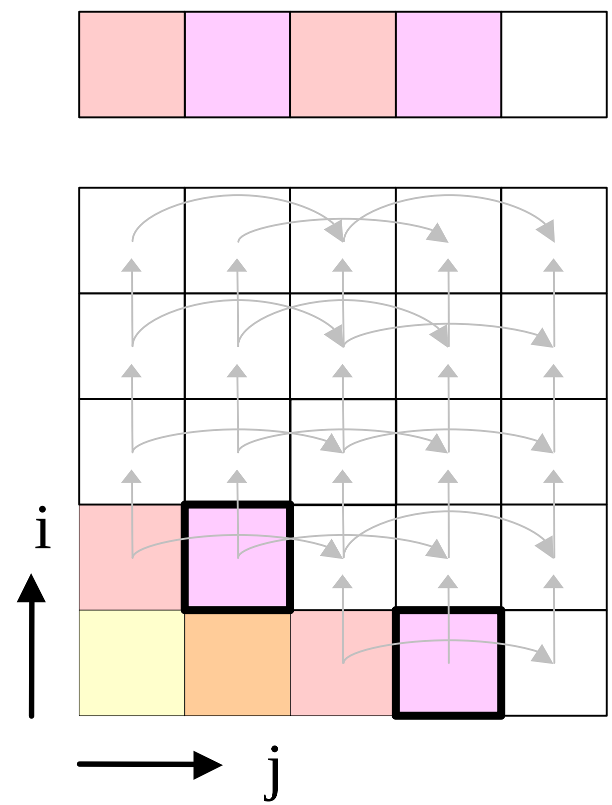

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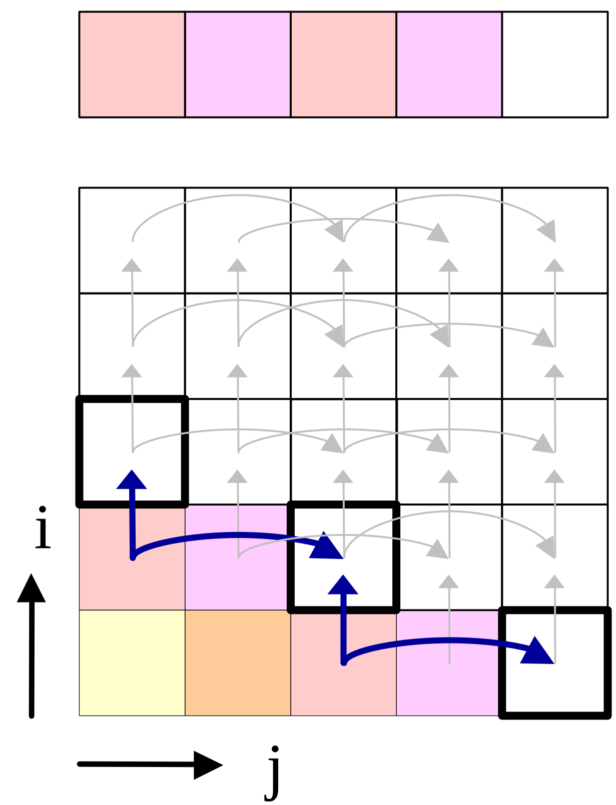

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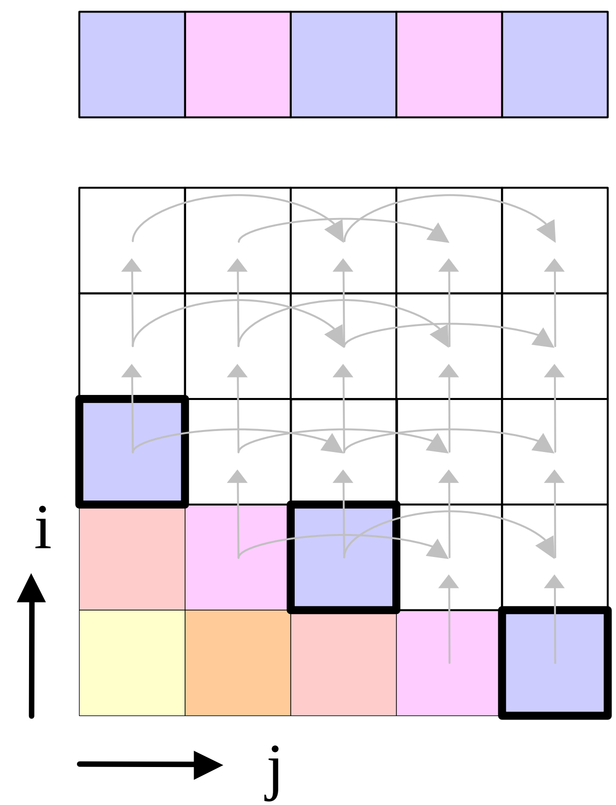

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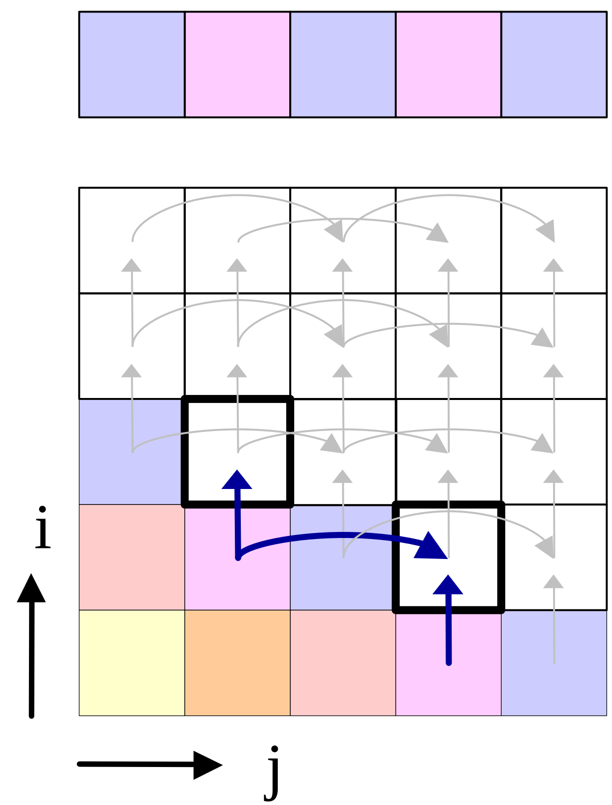

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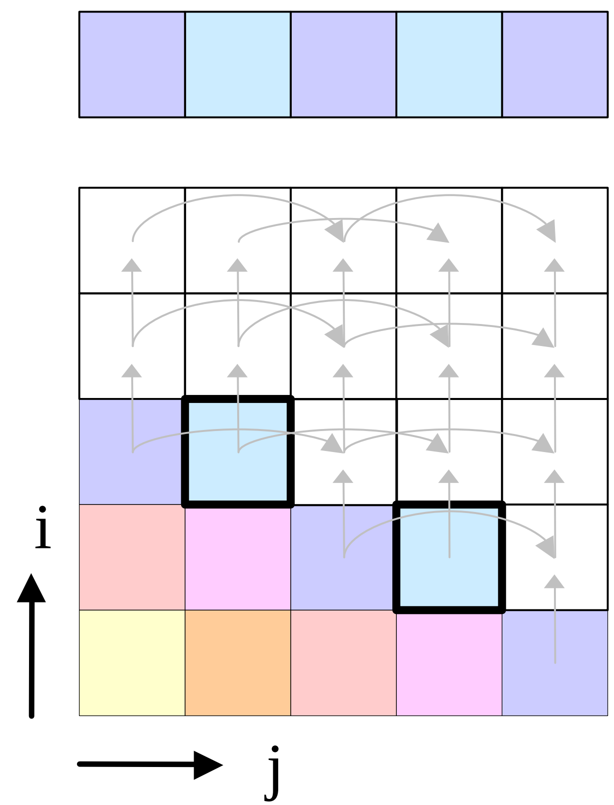

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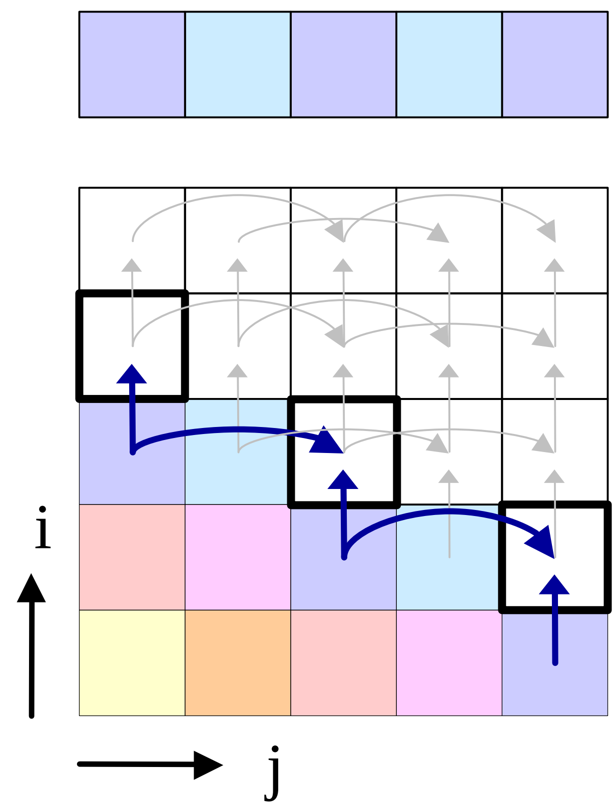
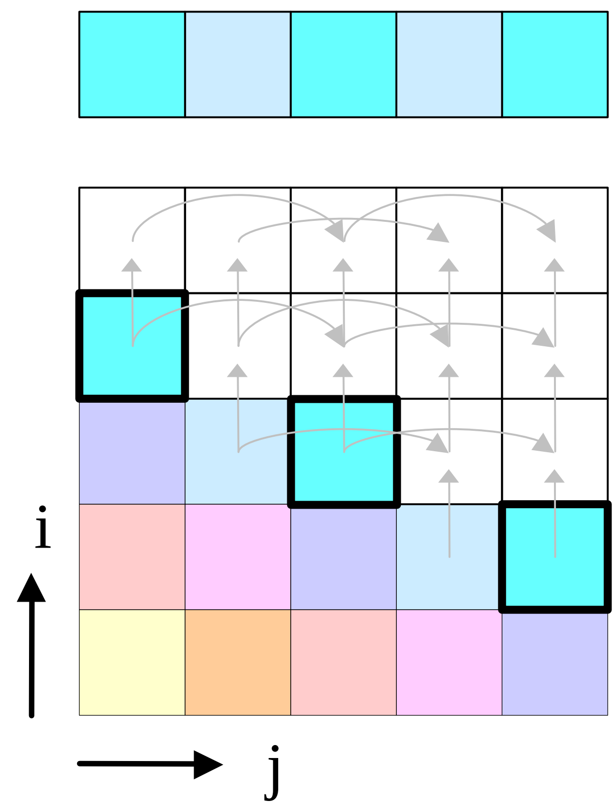


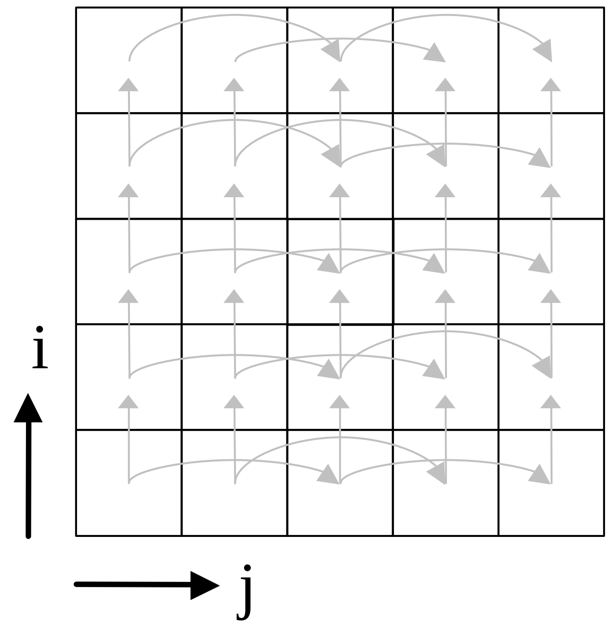
Answering Question #2

- Given $\vec{v} = (0, 1)$, what is the range of valid schedules θ?
  - ➔Lets try $\theta(i, j) = 2 * i + j$



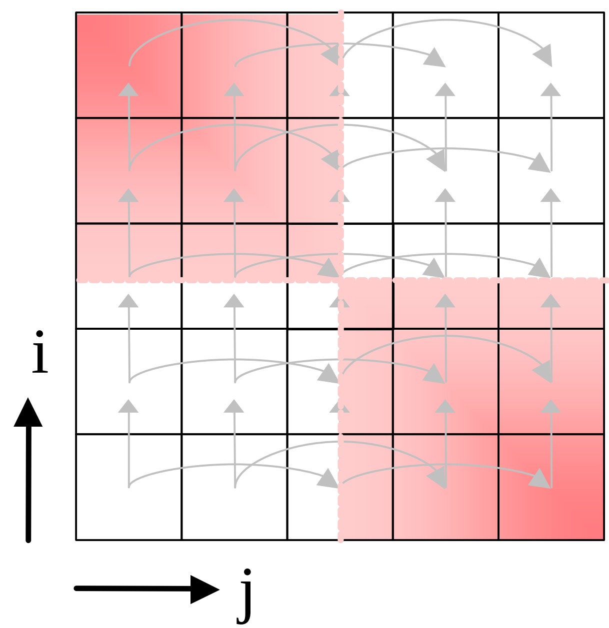
Answering Question #3

- What is the shortest $\vec{v}$ that is valid for all legal affine schedules?



Answering Question #3

- What is the shortest $\vec{v}$ that is valid for all legal affine schedules?
  - ➔Range of legal θ



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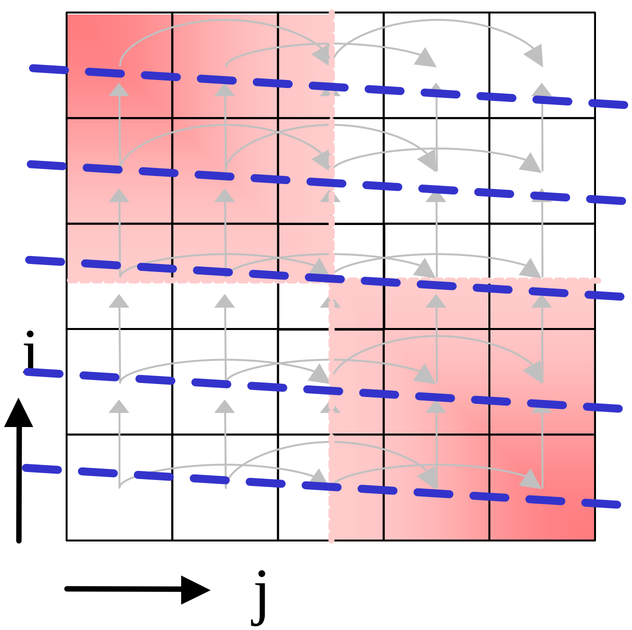

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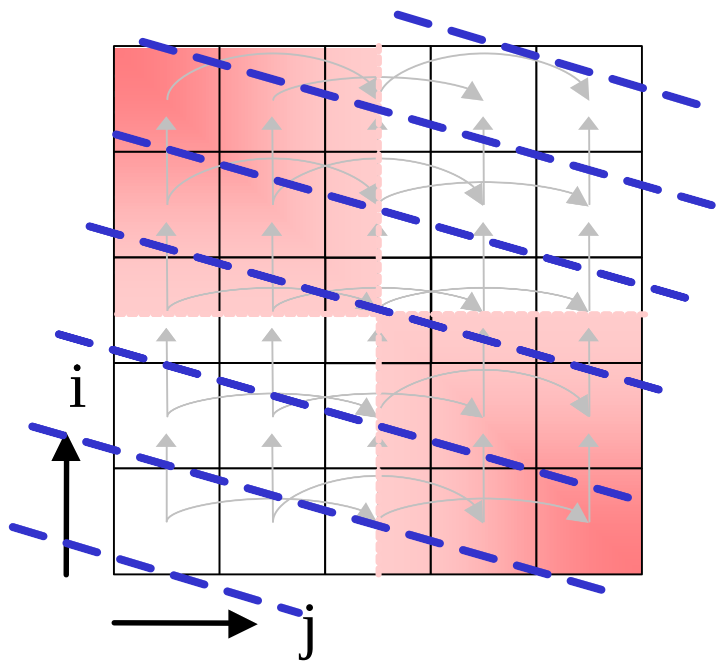

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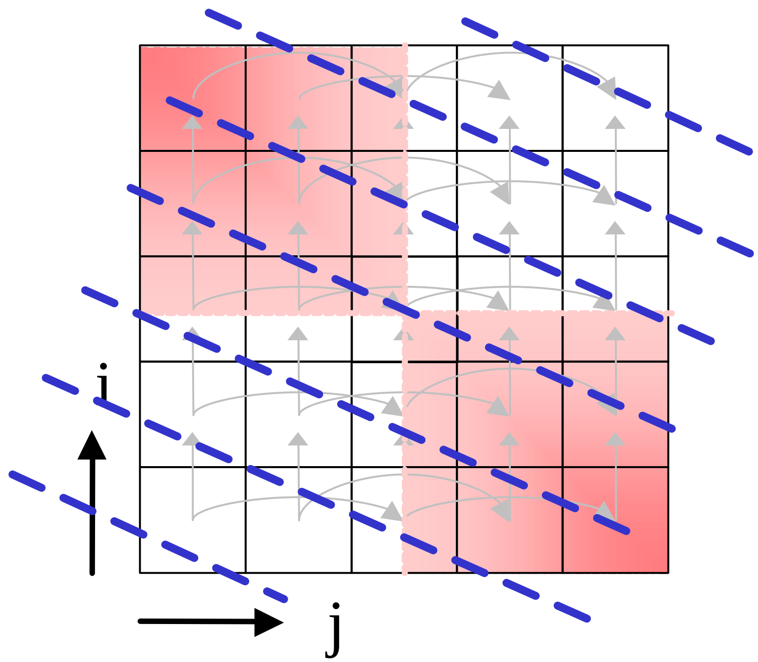

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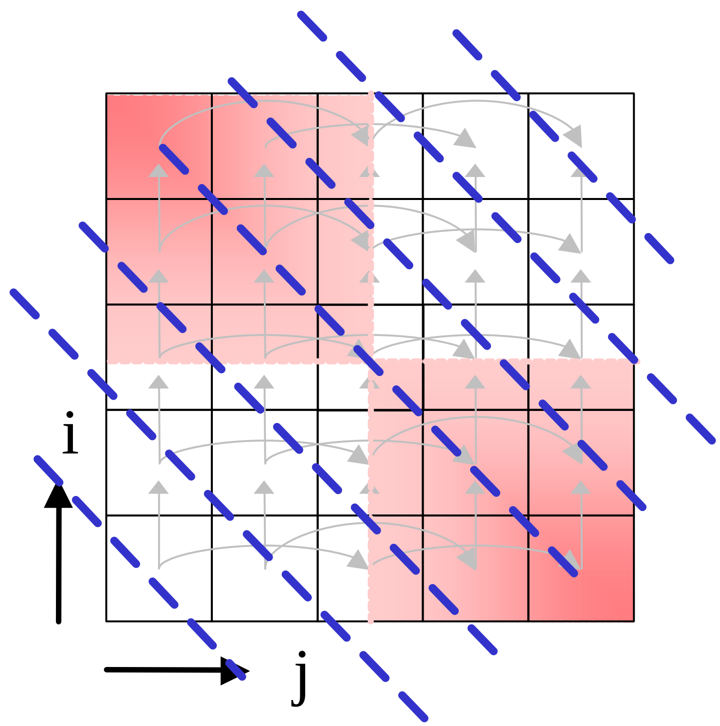

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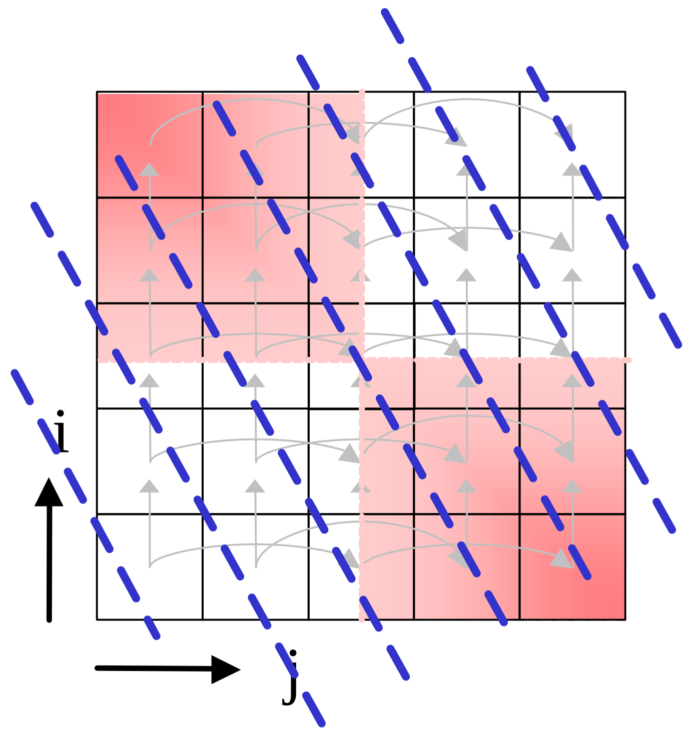

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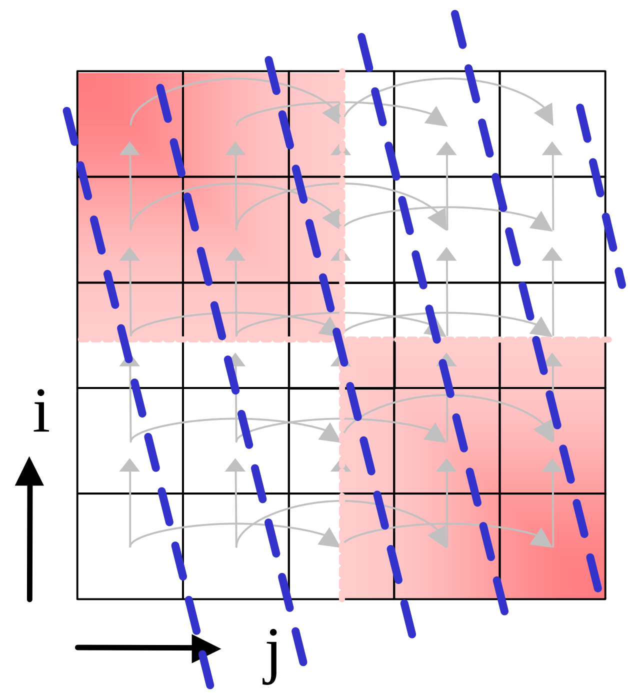

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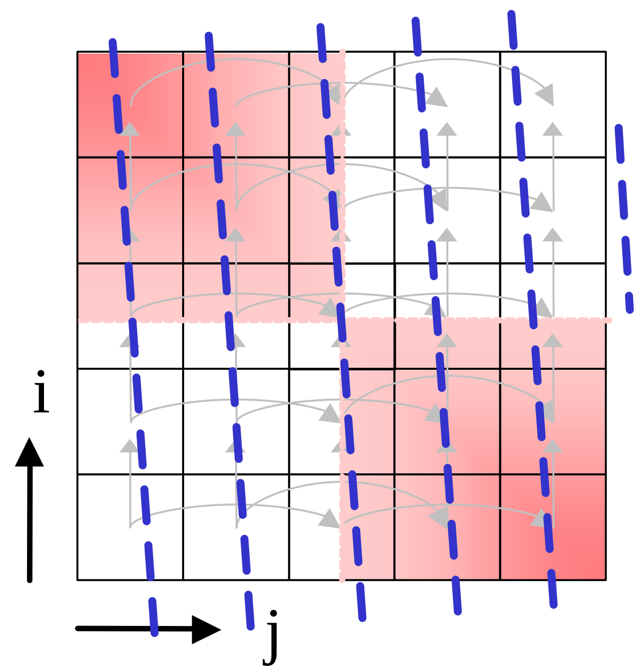

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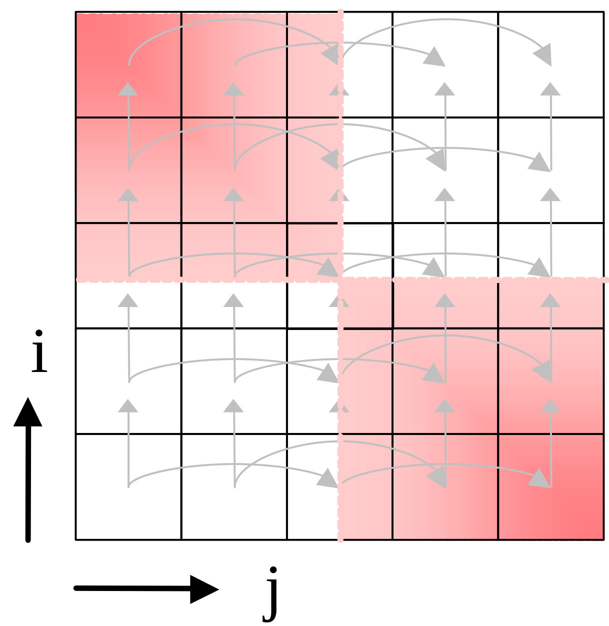
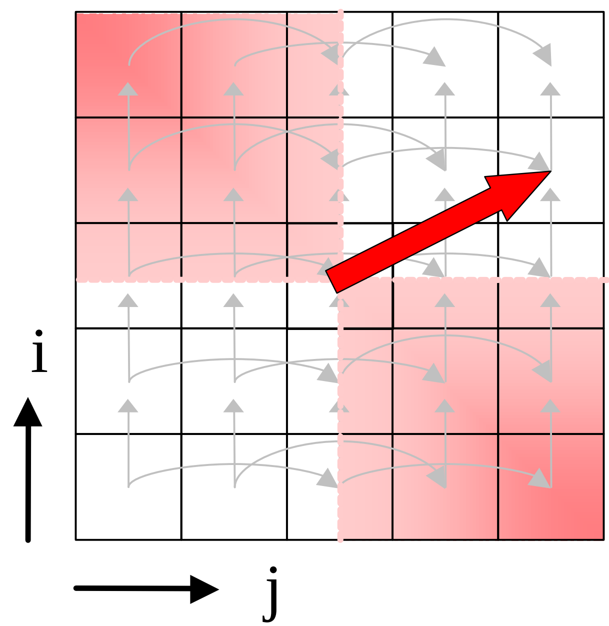


Answering Question #3

- What is the shortest $\vec{v}$ that is valid for all legal affine schedules?
  - ➔Range of legal θ
  - ➔$\vec{v}$ = (2, 1)



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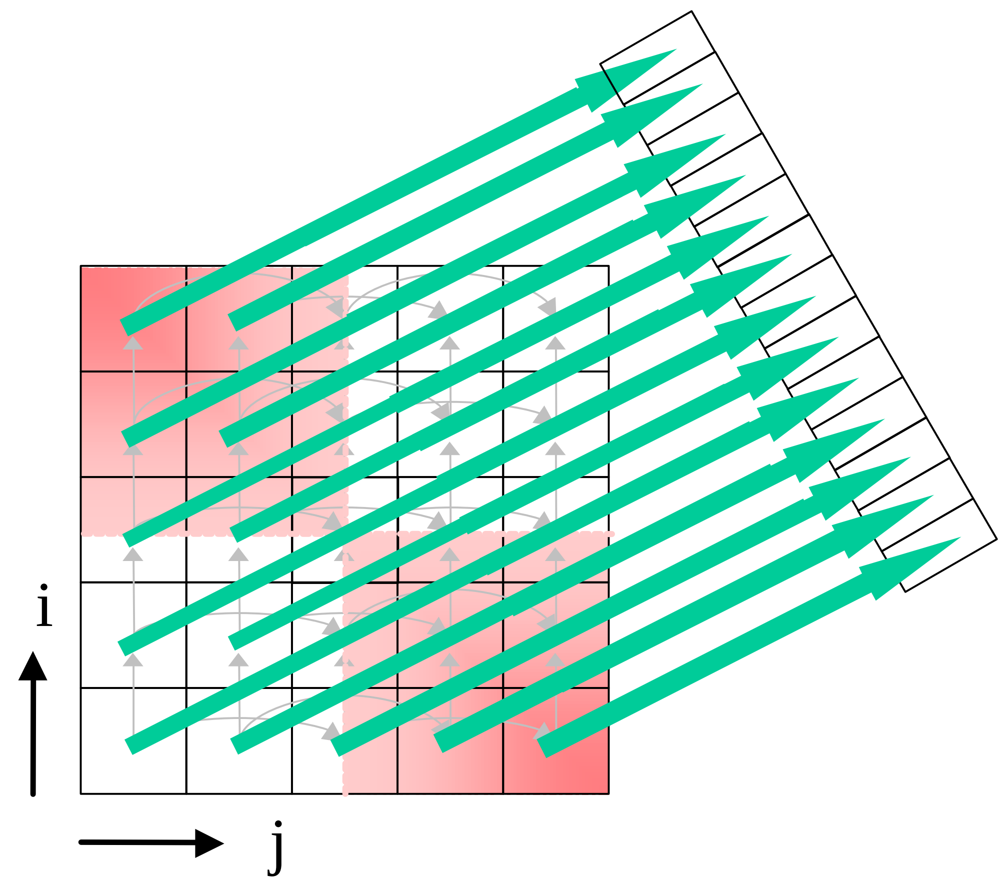

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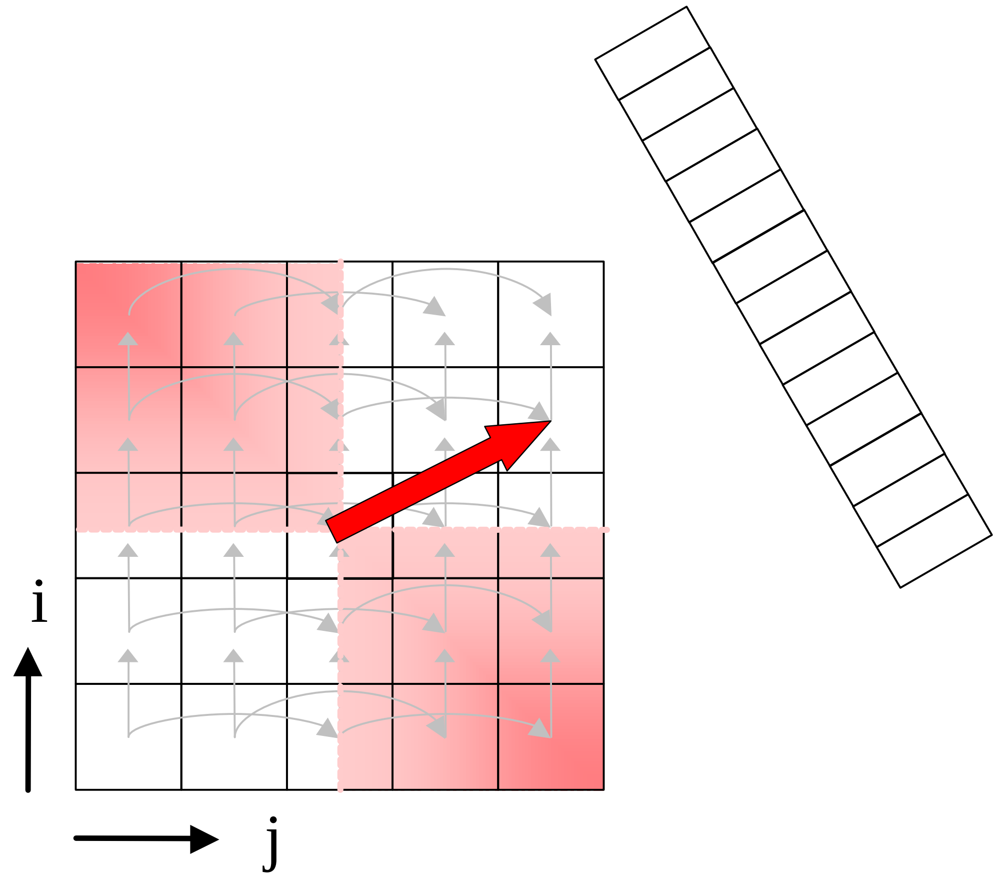

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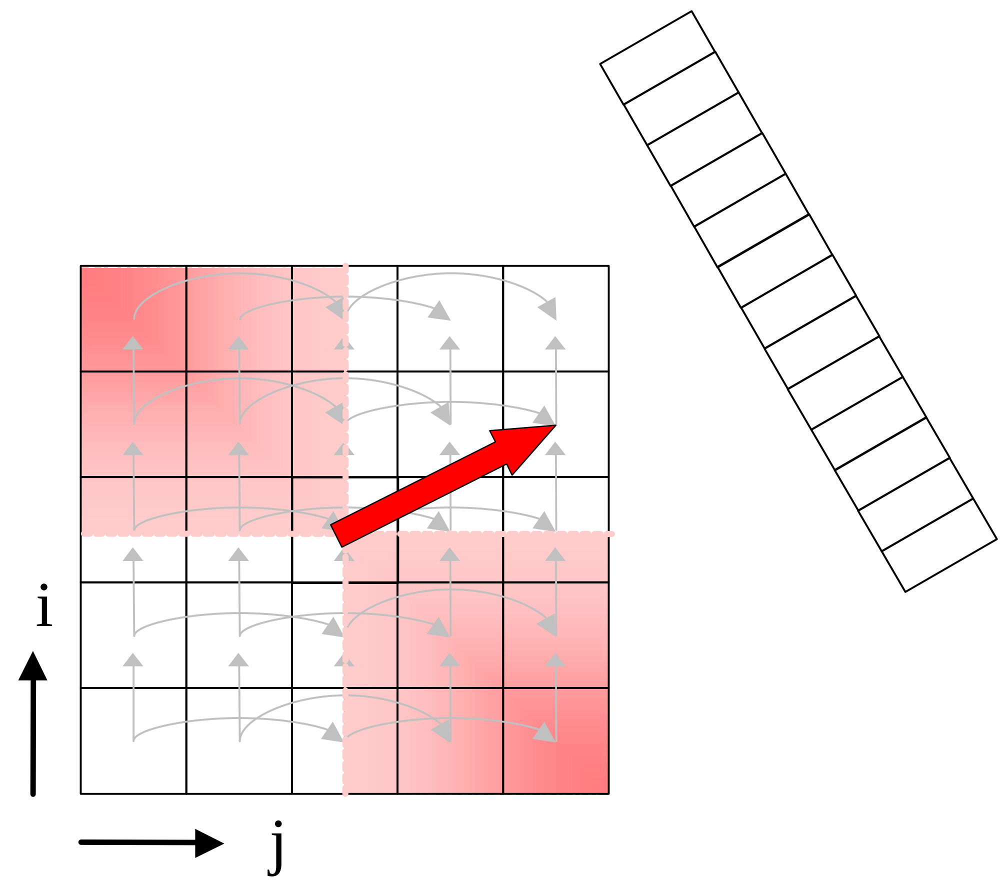


- <u>Def:</u> $\vec{v}$ is an affine occupancy vector (AOV)

Outline

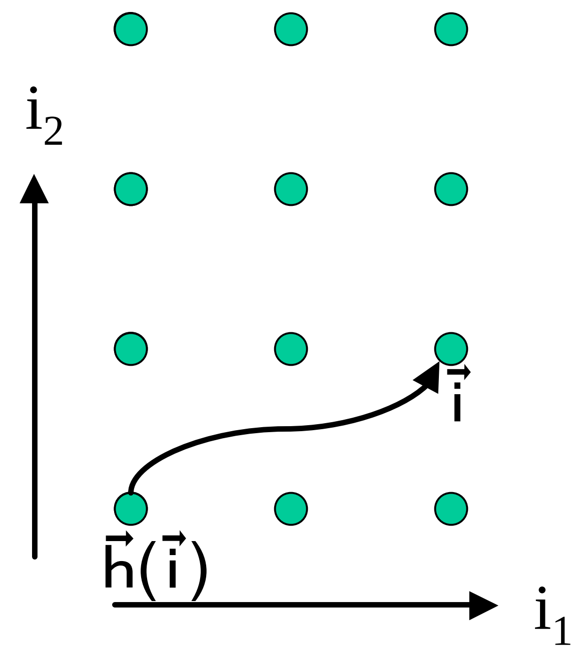
- Abstract problem
- Simplifications
- Concrete problem
- Solution Method
- Conclusions

Schedule Constraints

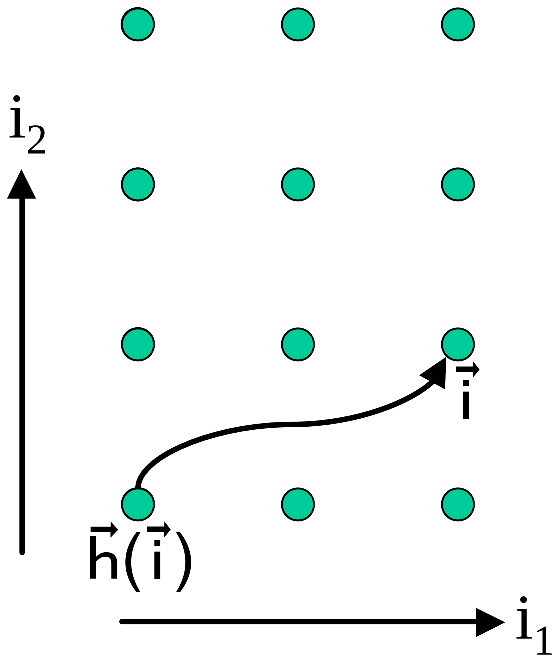
- Dependence analysis yields:
  - iteration $\vec{i}$ depends on iteration $\vec{h}(\vec{i})$
  - $\vec{h}$ is an affine function
- Consumer must execute after producer

Schedule Constraint

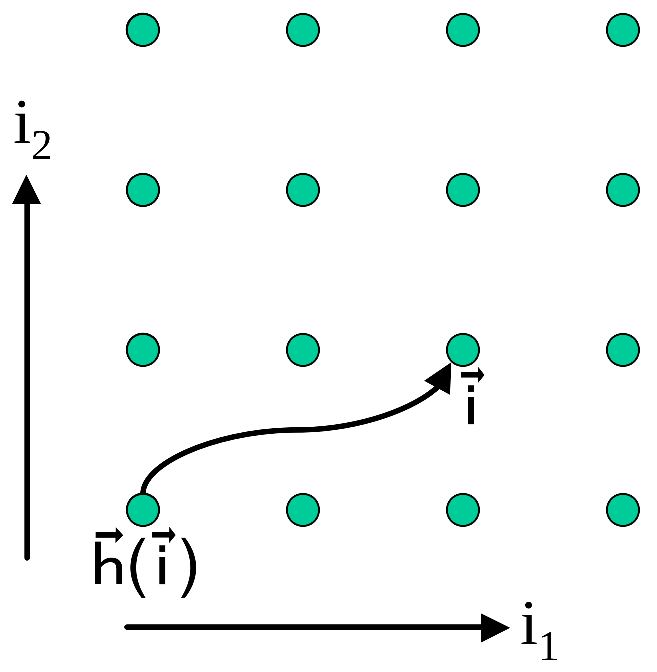
$$\theta(\vec{i}) \geq \theta(\vec{h}(\vec{i})) + 1$$

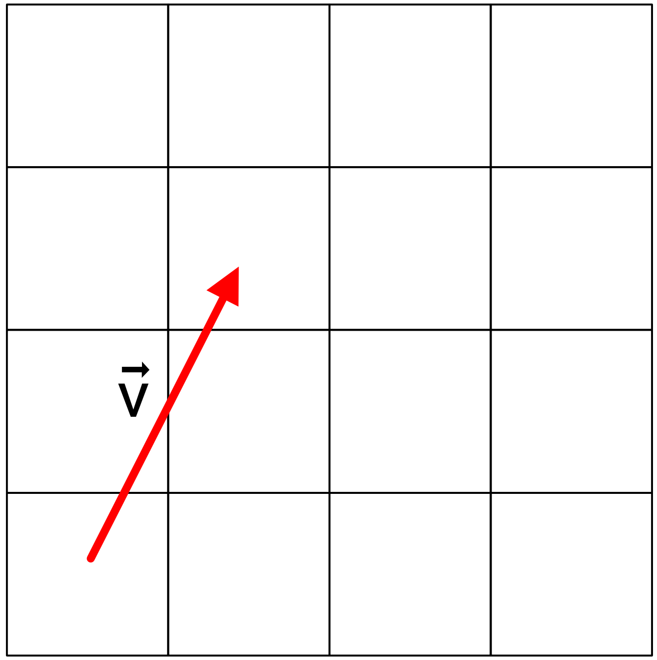


Storage Constraints



Storage Constraints





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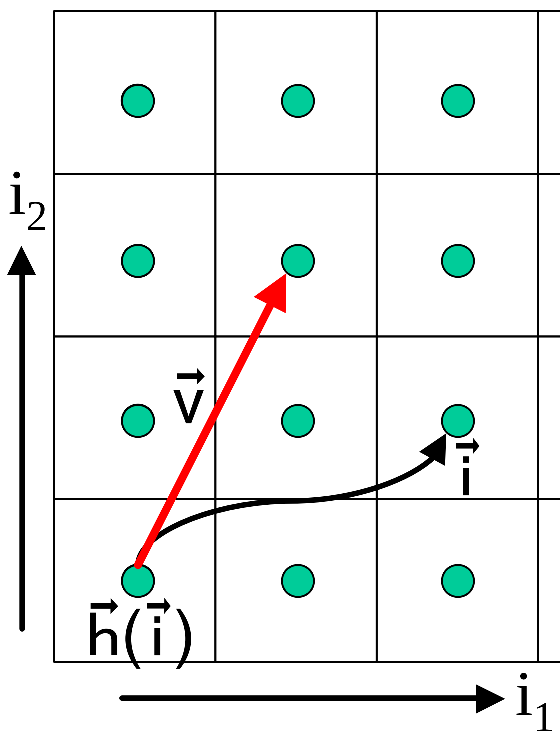

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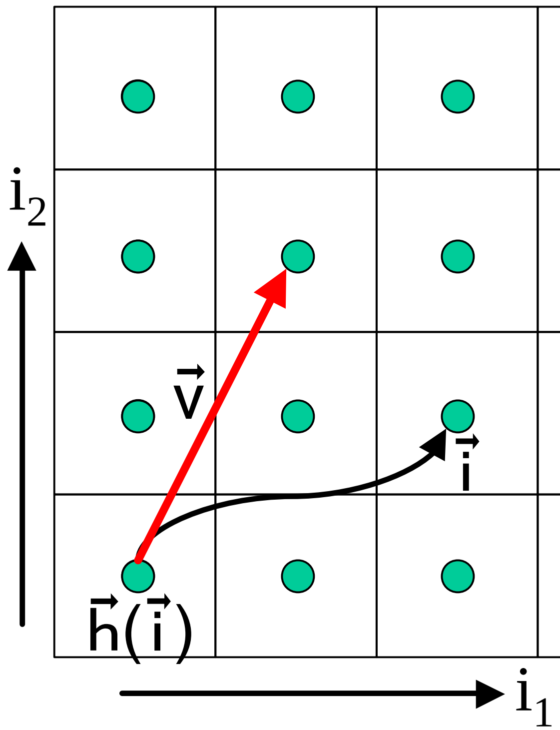
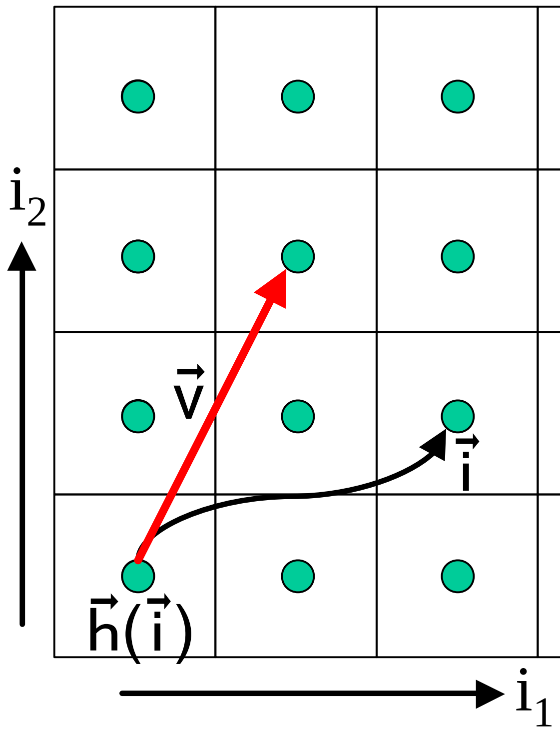


<u>dynamic single assignment</u>

```
for i = 1 to n
   for j = 1 to n
      A[i][j] = ...
      B[i][j] = ...
```

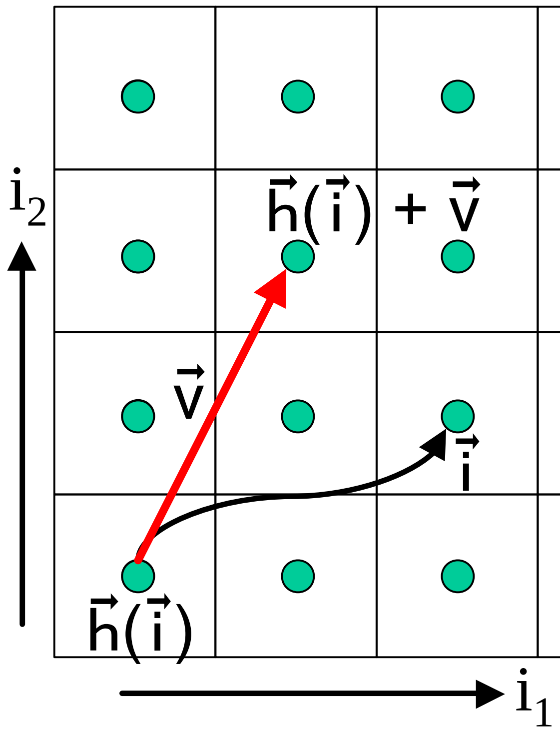
Storage Constraints



Consumer: $\vec{i}$

Producer: $\vec{h}(\vec{i})$

Storage Constraints

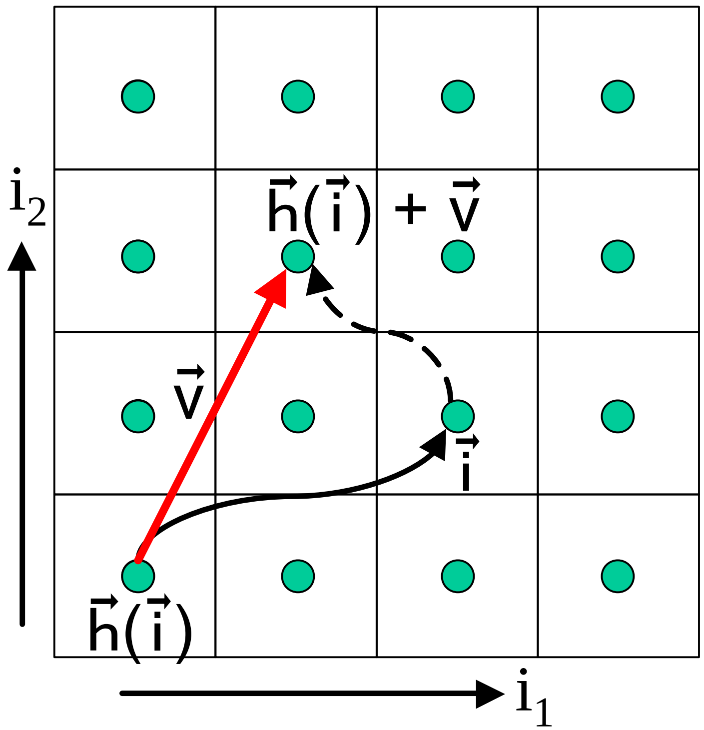


Consumer: $\vec{i}$

Producer: $\vec{h}(\vec{i})$

Overwriting producer: $\vec{h}(\vec{i}) + \vec{v}$

Storage Constraints



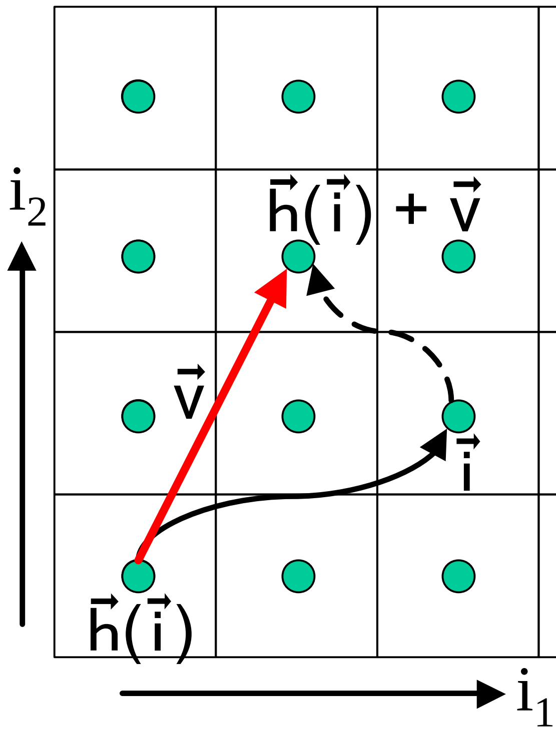
Consumer: $\vec{i}$

Producer: $\vec{h}(\vec{i})$

Overwriting producer: $\vec{h}(\vec{i}) + \vec{v}$

➔ Consumer must execute before producer is overwritten

Storage Constraints



Consumer: $\vec{i}$

Producer: $\vec{h}(\vec{i})$

Overwriting producer: $\vec{h}(\vec{i}) + \vec{v}$

- Consumer must execute before producer is overwritten

Storage Constraint

$$\theta(\vec{i}) \leq \theta(\vec{h}(\vec{i}) + \vec{v})$$

The Constraints

- A given $(\theta, \vec{v})$ combination is valid if
  - For all dependences $\vec{h}$,
  - For all iterations $\vec{i}$ in the program:

$$\begin{cases} \theta(\vec{i}) \geq \theta(\vec{h}(\vec{i})) + 1 & \text{schedule constraint} \\ \theta(\vec{i}) \leq \theta(\vec{h}(\vec{i}) + \vec{v}) & \text{storage constraint} \end{cases}$$

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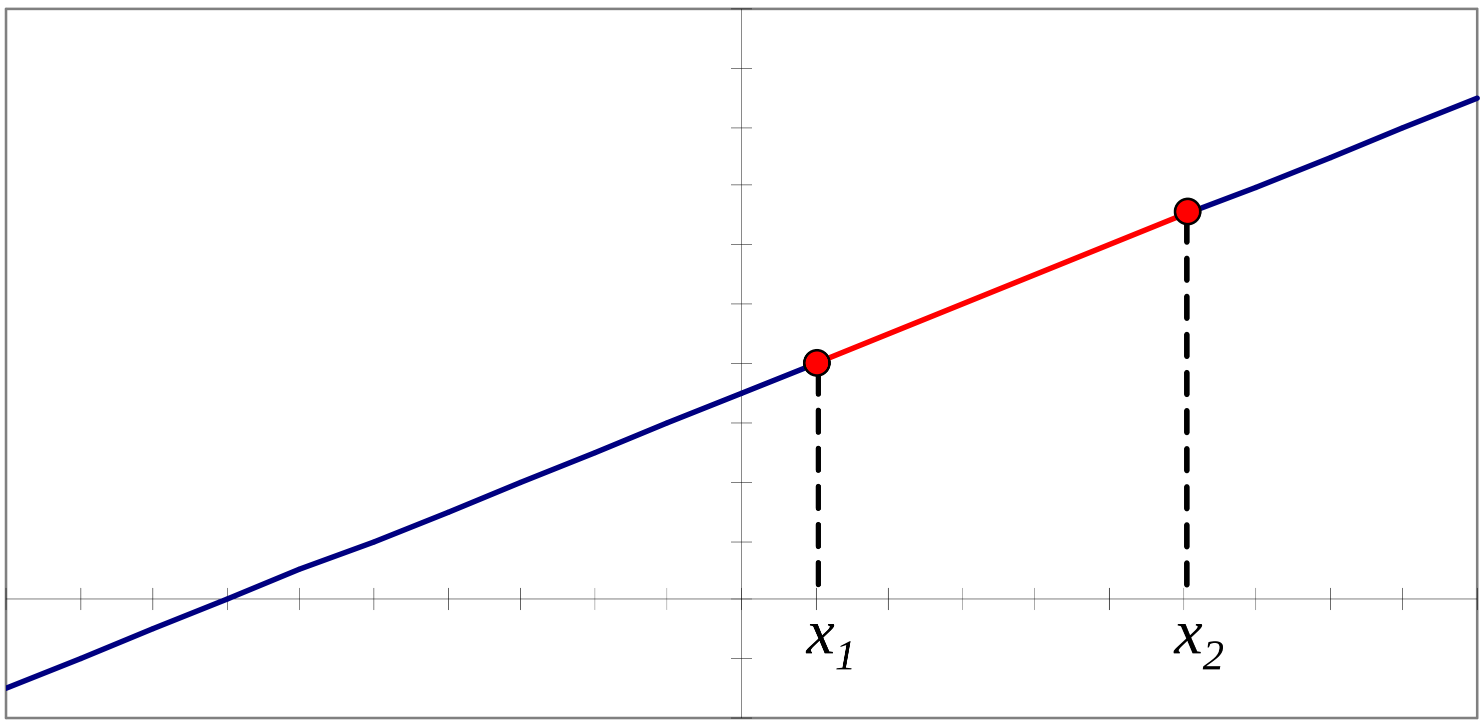
- Given θ, want to find $\vec{v}$ satisfying constraints
  - Might look simple, but it is not
  - Too many $\vec{i}$'s to enumerate!
  - Need to reduce the number of constraints

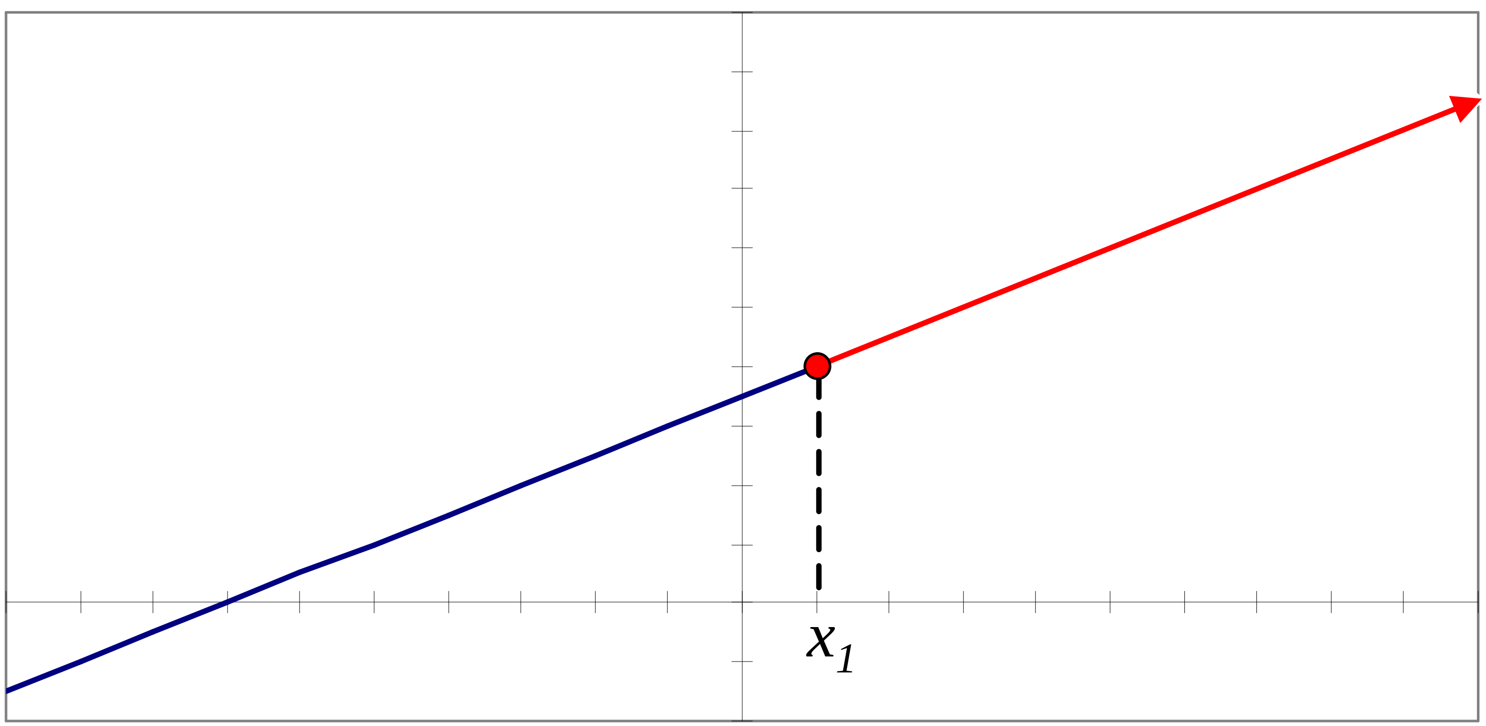
The Vertex Method (1-D)

- An affine function is non-negative within an interval $[x_1, x_2]$ iff it is non-negative at x_1 and x_2



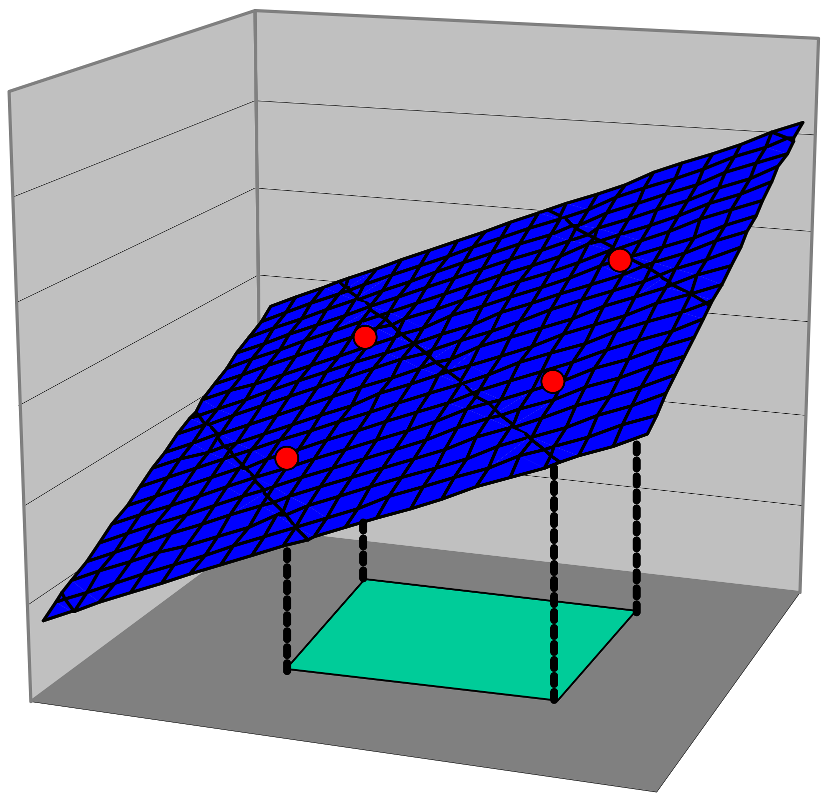
The Vertex Method (1-D)

- An affine function is non-negative over an unbounded interval $[x_1, \infty)$ iff it is non-negative at x_1 and is non-decreasing along the interval



The Vertex Method

- The same result holds in higher dimensions
  - An affine function is nonnegative over a bounded polyhedron D iff it is nonnegative at vertices of D



Applying the Method (Quinton87)

- Recall the storage constraints
  - For all iterations $\vec{i}$ in the program:

$$\theta(\vec{i}) \leq \theta(\vec{h}(\vec{i}) + \vec{v})$$

  - Re-arrange:

$$0 \leq \theta(\vec{h}(\vec{i}) + \vec{v}) - \theta(\vec{i})$$

- The right hand side is:
  1. An affine function of $\vec{i}$
  2. Nonnegative over the domain D of iterations

  ➔ We can apply the vertex method

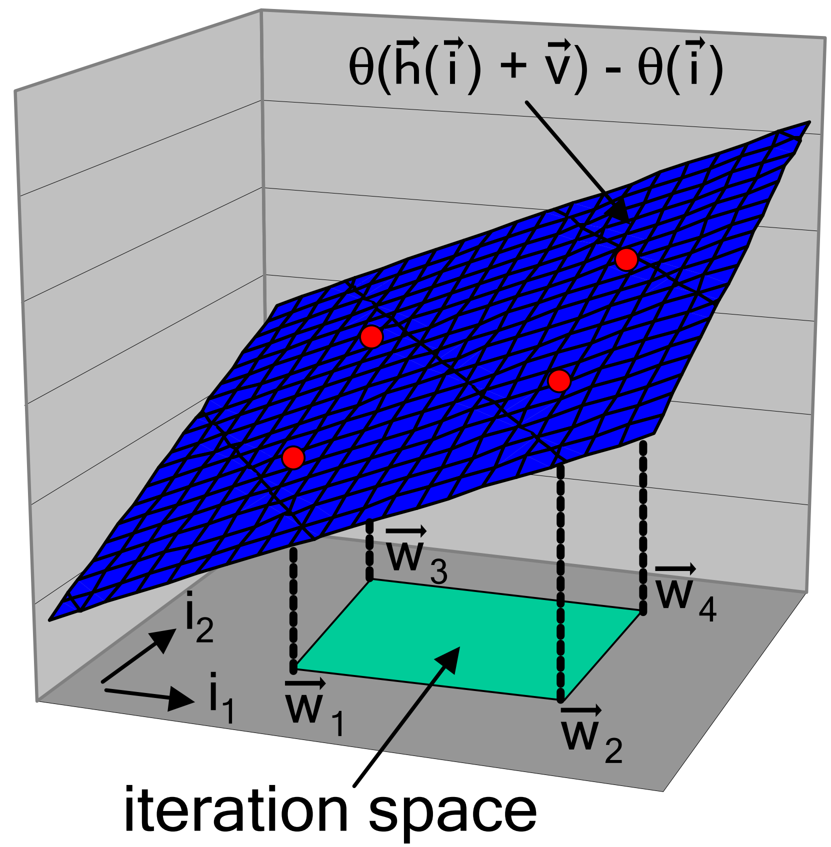
Applying the Method

- Replace $\vec{i}$ with the vertices $\overrightarrow{w}$ of its domain:

$$\forall \vec{i} \in D,\ \theta(\vec{h}(\vec{i}) + \vec{v}) - \theta(\vec{i}) \geq 0$$

$$\Downarrow$$

$$\begin{cases} \theta(\vec{h}(\overrightarrow{w}_1) + \vec{v}) - \theta(\overrightarrow{w}_1) \geq 0 \\ \theta(\vec{h}(\overrightarrow{w}_2) + \vec{v}) - \theta(\overrightarrow{w}_2) \geq 0 \\ \theta(\vec{h}(\overrightarrow{w}_3) + \vec{v}) - \theta(\overrightarrow{w}_3) \geq 0 \\ \theta(\vec{h}(\overrightarrow{w}_4) + \vec{v}) - \theta(\overrightarrow{w}_4) \geq 0 \end{cases}$$



The Reduced Constraints

- Apply same method to schedule constraints
- Now a given $(\theta, \vec{v})$ combination is valid if
  - For all dependences $\vec{h}$,
  - For all vertices $\vec{w}$ of the iteration domain:

$$\begin{cases} \theta(\vec{w}) \geq \theta(\vec{h}(\vec{w})) + 1 & \text{schedule constraint} \\ \theta(\vec{w}) \leq \theta(\vec{h}(\vec{w}) + \vec{v}) & \text{storage constraint} \end{cases}$$

- These are linear constraints
  - θ and $\vec{v}$ are variables; $\vec{h}$ and $\vec{w}$ are constants
  - Given θ, constraints are linear in $\vec{v}$ (& vice-versa)

Answering the Questions

$$\begin{cases} \theta(\vec{w}) \geq \theta(\vec{h}(\vec{w})) + 1 & \text{schedule constraint} \\ \theta(\vec{w}) \leq \theta(\vec{h}(\vec{w}) + \vec{v}) & \text{storage constraint} \end{cases}$$

1. Given θ, we can "minimize" $|\vec{v}|$
   - Linear programming problem
2. Given $\vec{v}$, we can find a "good" θ
   - Feautrier, 1992
3. To find an AOV... still too many constraints!
   - For all θ satisfying the schedule constraints:
     $\vec{v}$ must satisfy the storage constraints

Finding an AOV

$$\begin{cases} \theta(\vec{w}) \geq \theta(\vec{h}(\vec{w})) + 1 & \text{schedule constraint} \\ \theta(\vec{w}) \leq \theta(\vec{h}(\vec{w}) + \vec{v}) & \text{storage constraint} \end{cases}$$

- Apply the vertex method again!
  - Schedule constraints define domain of valid θ
  - Storage constraints can be written as a non-negative affine function of components of θ:
    - Expand $\theta(\vec{i}) = \vec{\beta} \cdot \vec{i}$

$$\vec{\beta} \cdot \vec{w} \leq \vec{\beta} \cdot (\vec{h}(\vec{w}) + \vec{v})$$

    - Simplify

$$(\vec{h}(\vec{w}) + \vec{v} - \vec{w}) \cdot \vec{\beta} \geq 0$$

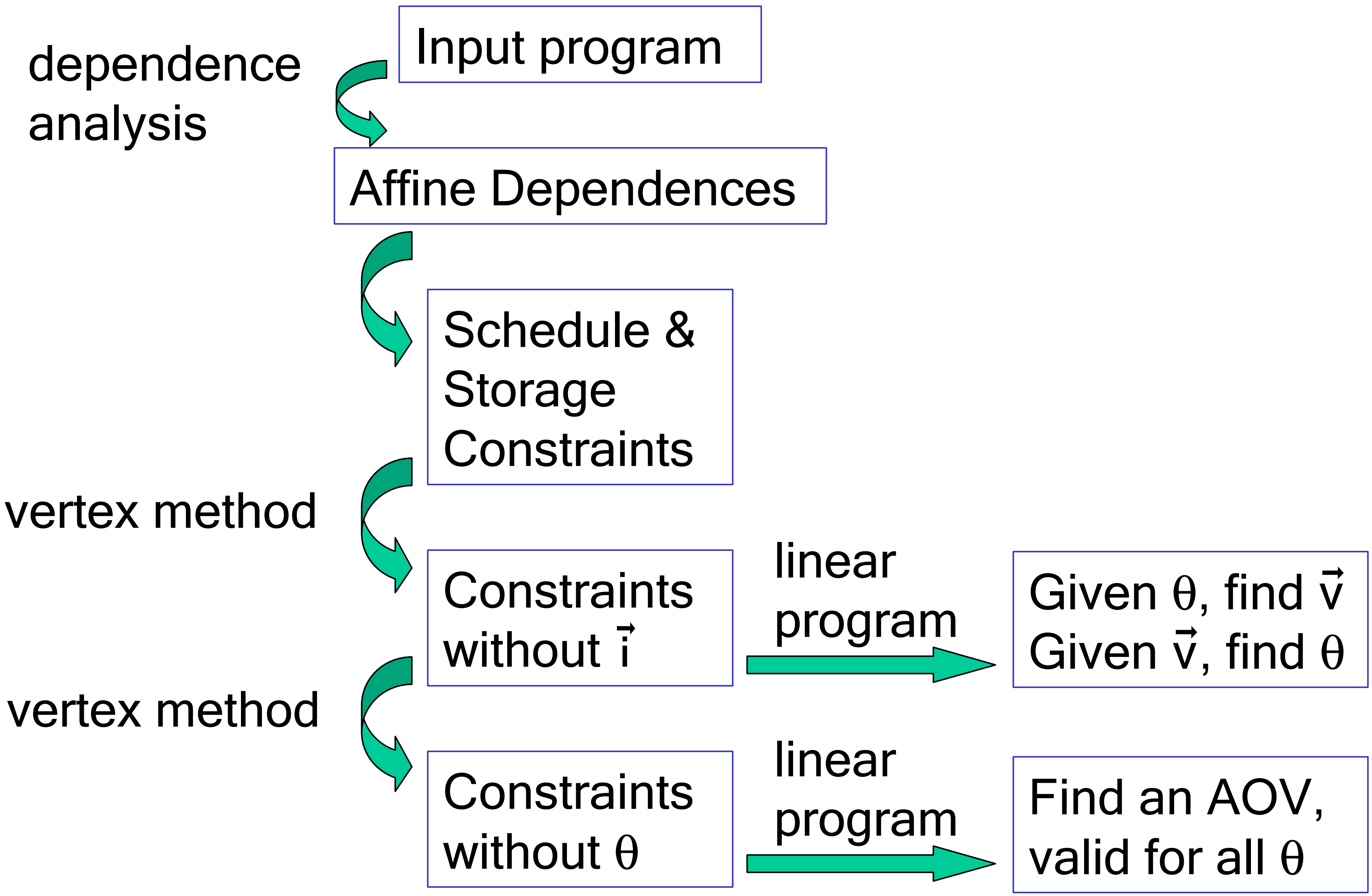
Finding an AOV

- Our constraints are now as follows:
  - For all dependences $\vec{h}$,
  - For all vertices $\vec{w}$ of the iteration domain,
  - For all vertices $\vec{\tau}$ of the space of valid schedules:

$$\vec{\tau} \bullet \vec{w} \leq \vec{\tau} \bullet (\vec{h}(\vec{w}) + \vec{v}) \qquad \text{AOV constraint}$$

- Can find "shortest" AOV with linear program
  - Finite number of constraints
  - $\vec{h}$, $\vec{w}$, and $\vec{\tau}$ are known constants

The Big Picture



Details in Paper

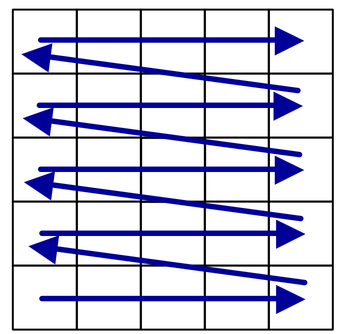
- Symbolic constants
- Inter-statement dependences across loops
- Farkas' Lemma for improved efficiency

Related Work

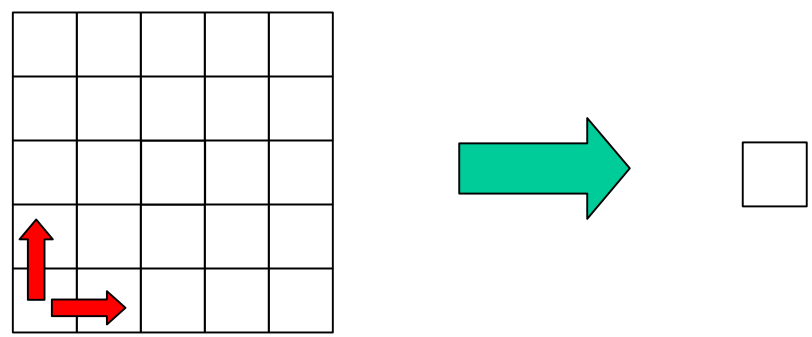
- Universal Occupancy Vector (Strout et al.)
  - Valid for all schedules, not just affine ones
  - Stencil of dependences in single loop nest
- Storage for ALPHA programs (Quilleré, Rajopadhye, Wilde)
  - Polyhedral model, with occupancy vector analog
  - Assume schedule is given
- PAF compiler (Cohen, Lefebvre, Feautrier)
  - Minimal expansion $\rightarrow$ scheduling $\rightarrow$ contraction
  - Storage mapping A[i mod x][j mod y]

Future Work

- Allow affine left hand side references
  - A[2∗j][n-i] = …
- Consider multi-dimensional time schedules



- Collapse multiple dimensions of storage



Conclusions

- Unified framework for determining:
  1. A good storage mapping for a given schedule
  2. A good schedule for a given storage mapping
  3. A good storage mapping for all valid schedules
- Take away: representations and techniques
  - Occupancy vectors
  - Affine schedules
  - Vertex method